

THE GALOIS MODULE STRUCTURE OF ℓ -ADIC REALIZATIONS OF PICARD 1-MOTIVES AND APPLICATIONS

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ABSTRACT. Let $Z \rightarrow Z'$ be a G -Galois cover of smooth, projective curves over an arbitrary algebraically closed field κ , and let \mathcal{S} and \mathcal{T} be G -equivariant, disjoint, finite, non-empty sets of closed points on Z , such that \mathcal{S} contains the ramification locus of the cover. In this context, we prove that the ℓ -adic realizations $T_\ell(\mathcal{M}_{\mathcal{S},\mathcal{T}})$ of the Picard 1-motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}$ associated to the data $(Z, \kappa, \mathcal{S}, \mathcal{T})$ are G -cohomologically trivial and therefore $\mathbb{Z}_\ell[G]$ -projective modules of finite rank, for all prime numbers ℓ . In the process, we give a new proof of Nakajima's theorem [16] on the Galois module structure of the semi-simple piece $\Omega_Z(-[\mathcal{S}])^s$ under the action of the Cartier operator on a certain space of differentials $\Omega_Z(-[\mathcal{S}])$ associated to Z and \mathcal{S} , assuming that $\text{char}(\kappa) = p$. As a main arithmetic application of these results, we consider the situation where the set of data $(Z, Z', \mathcal{S}, \mathcal{T})$ is defined over a finite field \mathbb{F}_q and $\kappa := \overline{\mathbb{F}_q}$. We combine results of Deligne and Tate, Berthelot, Bloch and Illusie with our cohomological triviality result to prove that in this context we have an equality of $\mathbb{Z}_\ell[[\overline{G}]]$ -ideals $(\Theta_{\mathcal{S},\mathcal{T}}(\gamma^{-1})) = \text{Fit}_{\mathbb{Z}_\ell[[\overline{G}]]}(T_\ell(\mathcal{M}_{\mathcal{S},\mathcal{T}}))$, for all prime numbers ℓ , where $\overline{G} := G \times \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, γ is the q -power arithmetic Frobenius morphism (viewed as a distinguished topological generator of $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$) and $\Theta_{\mathcal{S},\mathcal{T}}(u) \in \mathbb{Z}[G][u]$ is the (polynomial) G -equivariant L -function associated to the data $(Z, Z', \mathbb{F}_q, \mathcal{S}, \mathcal{T})$. We obtain this way a Galois-equivariant refinement of results of Deligne and Tate [20] on ℓ -adic realizations of Picard 1-motives associated to (global) function fields. As an immediate application, we prove refinements of the classical Brumer-Stark and Coates-Sinnott conjectures linking special values of $\Theta_{\mathcal{S},\mathcal{T}}(u)$ to certain invariants of ideal-class groups and étale cohomology groups, respectively. In our upcoming work, we will show how several other classical conjectures on special values of global L -functions follow from the results obtained in this paper.

1. Introduction

The main goal of this paper is twofold. *First*, we consider a Galois cover $Z \rightarrow Z'$ of Galois group G of smooth, projective curves over an arbitrary algebraically closed field κ , and two G -invariant, disjoint, finite, non-empty sets of closed points \mathcal{S} and \mathcal{T} on Z , such that \mathcal{S} contains the ramification locus of the cover. In this context, we prove that the ℓ -adic realizations $T_\ell(\mathcal{M}_{\mathcal{S},\mathcal{T}})$ of the Picard 1-motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}$ associated to the data $(Z, \kappa, \mathcal{S}, \mathcal{T})$ are G -cohomologically trivial and therefore $\mathbb{Z}_\ell[G]$ -projective modules of finite rank, for all prime numbers ℓ (see Theorem 3.10.)

This goal is achieved in two main steps: in the first, we give a new interpretation of the groups $\mathcal{M}_{\mathcal{S},\mathcal{T}}[n]$ of n -torsion points of $\mathcal{M}_{\mathcal{S},\mathcal{T}}$ (see Proposition 2.9); in the second step, we use this interpretation to give a complete description of the $\mathbb{F}_\ell[G]$ -module structure of $\mathcal{M}_{\mathcal{S},\mathcal{T}}[\ell]$ in the case where G is an ℓ -group. We prove that this module is free and explicitly compute its rank in terms of the

2000 *Mathematics Subject Classification.* 11M38, 11G20, 11G25, 11G45, 14F30.

Key words and phrases. Picard 1-motives, global L -functions, étale and crystalline cohomology, Galois module structure, Fitting ideals.

The second author was partially supported by NSF Grants DMS-0600905 and DMS-0901447.

genus of Z' and the Hasse-Witt invariant of Z' , depending on whether $\ell \neq \text{char}(\kappa)$ or $\ell = \text{char}(\kappa)$, respectively (see Theorem 3.6.) This result implies our first main Theorem 3.10.

Another immediate application of Proposition 2.9 and Theorem 3.6 is a new proof of a classical theorem of Nakajima [16] on the Galois module structure of the semi-simple piece $\Omega_Z(-[\mathcal{S}])^s$ under the action of the Cartier operator on a certain space of differentials $\Omega_Z(-[\mathcal{S}])$ associated to Z and \mathcal{S} , assuming that $\text{char}(\kappa) = p$. In §6, we show that Nakajima's theorem is a particular case of our Theorem 3.6, placing this way Nakajima's results in the general context of Picard 1-motives (see Proposition 6.7.) In upcoming work [7], we will apply Proposition 2.9 to give explicit formulas for the canonical perfect duality pairings between the torsion (or ℓ -adic realizations) of the Picard 1-motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}$ and its dual (Albanese) 1-motive $\mathcal{M}_{\mathcal{T},\mathcal{S}}$, generalizing the classical formulas for the Weil pairings at the level of Jacobians of curves.

Second, we focus our attention on the arithmetically interesting situation where G is abelian, the set of data $(Z, Z', \mathcal{S}, \mathcal{T})$ is defined over a finite field \mathbb{F}_q of characteristic p (in the sense explained below) and $\kappa = \mathbb{F}$ (the algebraic closure of \mathbb{F}_q .) In this case, we have a G -Galois cover $Z_0 \rightarrow Z'_0$ of smooth projective curves over \mathbb{F}_q , $Z = Z_0 \times_{\mathbb{F}_q} \mathbb{F}$ and $Z' = Z'_0 \times_{\mathbb{F}_q} \mathbb{F}$, S and T are two finite, disjoint sets of closed points on Z'_0 , S contains the ramified locus of the cover, and \mathcal{S} and \mathcal{T} consist of all closed points on Z sitting above points in S and T , respectively. To the set of data $(Z_0, Z'_0, \mathbb{F}_q, S, T)$ one can associate a polynomial G -equivariant L -function $\Theta_{S,T}(u) \in \mathbb{Z}[G][u]$, obtained via an equivariant construction from the classical Artin L -functions for the G -cover of \mathbb{F}_q -schemes $Z_0 \rightarrow Z'_0$ (see Chpt. V of [20] and §4 below.) Deligne expresses $\Theta_{S,T}(u)$ in terms of the (G -equivariant) characteristic polynomial of the q -power geometric Frobenius morphism acting on the \mathbb{Q}_ℓ -representation $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(\mathcal{M}_{\mathcal{S},\mathcal{T}})$ of G , for any prime $\ell \neq p$ (see [20], Chpt. V.) One can combine Deligne's result with a theorem of Berthelot and express $\Theta_{S,T}(u)$ in terms of the G -equivariant q -power Frobenius action on the crystalline cohomology group $H^1_{\text{cris}}(Z/\mathbb{C}_p)$ (see §4 below and the Appendix of [17].) In §4, we combine this link with a result of Bloch and Illusie relating crystalline and p -adic étale cohomology and with our cohomological triviality result (Theorem 3.10) to show that we have an equality of $\mathbb{Z}_\ell[[\overline{G}]]$ -ideals

$$(\Theta_{S,T}(\gamma^{-1})) = \text{Fit}_{\mathbb{Z}_\ell[[\overline{G}]]}(T_\ell(\mathcal{M}_{\mathcal{S},\mathcal{T}})),$$

for all prime numbers ℓ , where $\overline{G} := G \times \text{Gal}(\mathbb{F}/\mathbb{F}_q)$, γ is the q -power arithmetic Frobenius viewed as a distinguished topological generator of $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$, and “Fit” stands for the first Fitting ideal (see Theorem 4.3.) As an immediate consequence of Theorem 4.3, in §5 we prove refined versions of the function field analogues of the classical Brumer-Stark and Coates-Sinnott conjectures, linking special values of the equivariant L -function $\Theta_{S,T}$ to Galois-module structure invariants of certain ideal class-groups and étale cohomology groups of K , respectively (see Theorems 5.18 and 5.20.)

In upcoming work, we show how our main Theorems 3.10 and 4.3 permit us to prove (or, in some cases, give new proofs of) several other classical conjectures on special values of L -functions for characteristic p global fields (function fields), e.g. the Rubin-Stark and Gross conjectures. Other applications of our work on Tate modules of Picard 1-motives include explicit calculations of Fitting ideals over Galois-group rings of (Pontrjagin duals of) ideal-class groups (see [8]) as well as a possible new way of looking at Tate sequences and canonical classes. Further, we are in the process of developing a number field Iwasawa theoretic analogue of this theory with similar results and applications, under certain restrictive hypotheses. Just as our work over finite fields can be viewed as a Galois-equivariant refinement of work of Deligne and Tate [20] on ℓ -adic realizations of Picard 1-motives associated to (global) function fields, its intended number field analogue will be

a natural Galois-equivariant refinement of Wiles's results [21] on the Main Conjecture in classical Iwasawa Theory over totally real number fields.

Acknowledgement. The authors would like to thank their home universities for making mutual visits possible. These visits were funded by the DFG and the NSF, respectively. The second author would like to thank Kiran Kedlaya for fruitful discussions and especially for bringing Nakajima's work [16] to his attention.

2. Picard 1-motives, their torsion points and ℓ -adic realizations

In this section, we recall the definitions and main properties of a special class of 1-motives called Picard 1-motives, and briefly review their torsion structure and ℓ -adic realizations. The section ends with a proposition which gives a new interpretation of the group of n -torsion points of a Picard 1-motive, which will prove very useful in our future considerations. For general properties of 1-motives the reader can consult Chpt. 10 of Deligne's paper [6]. For the arithmetic situation treated in the last section of this paper, the reader will find Chpt. V of [20] useful as well.

Let Z denote a smooth, projective (not necessarily connected) curve over an algebraically closed field κ . Let $\mathcal{K} := \kappa(Z)$ be the κ -algebra of rational functions on Z . If Z is connected, then \mathcal{K} is a field; otherwise, \mathcal{K} is a finite direct sum of fields (finitely generated, of transcendence degree 1 over κ). These are the fields of rational functions on the connected components of Z . We let \mathcal{S} and \mathcal{T} denote two finite (possibly empty), disjoint sets of closed points on Z .

Definition 2.1. For \mathcal{S} and \mathcal{T} as above and all $n \in \mathbb{N}$, we define the following subgroups of the multiplicative group \mathcal{K}^\times of invertible elements in \mathcal{K} .

$$\mathcal{K}_\mathcal{T}^\times := \{f \in \mathcal{K}^\times \mid f(P) = 1, \forall P \in \mathcal{T}\},$$

$$\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)} := \{f \in \mathcal{K}_\mathcal{T}^\times \mid \text{div}(f) = nD + y\},$$

where D is an arbitrary divisor on Z and y is a divisor on Z supported on \mathcal{S} .

If \mathcal{R} is a (possibly infinite) set of closed points on Z , we denote by $\text{Div}^0(\mathcal{R})$ the abelian group of divisors supported on \mathcal{R} and of *multidegree* 0 (i.e. of degree 0 when restricted to each of the connected components of Z .) In what follows, we will denote by J the Jacobian variety associated to Z . We remind the reader that J is a connected abelian variety defined over κ of dimension equal to the sum of the genera of the connected components of Z . If $J(\kappa)$ denotes the group of κ -rational points of J , then we have a well-known canonical group isomorphism

$$J(\kappa) \xrightarrow{\sim} \frac{\text{Div}^0(Z)}{\{\text{div}(f) \mid f \in \mathcal{K}^\times\}}.$$

The generalized Jacobian $J_\mathcal{T}$ associated to (Z, \mathcal{T}) is a semi-abelian variety (an extension of the abelian variety J by a torus $\tau_\mathcal{T}$). At the level of κ -rational points, there are canonical group isomorphisms

$$\tau_\mathcal{T}(\kappa) \xrightarrow{\sim} \bigoplus_{z \in \pi_0(Z)} \frac{\bigoplus_{v \in \mathcal{T}_z} \kappa^\times}{\kappa^\times}, \quad J_\mathcal{T}(\kappa) \xrightarrow{\sim} \frac{\text{Div}^0(Z \setminus \mathcal{T})}{\{\text{div}(f) \mid f \in \mathcal{K}_\mathcal{T}^\times\}},$$

where $\pi_0(Z)$ denotes the set of connected components of Z , \mathcal{T}_z denotes the set of those points in \mathcal{T} which lie on z , and κ^\times sits inside $\bigoplus_{v \in \mathcal{T}_z} \kappa^\times$ diagonally, for all $z \in \pi_0(Z)$. The isomorphisms above lead to a short exact sequence of groups

$$0 \longrightarrow \tau_\mathcal{T}(\kappa) \longrightarrow J_\mathcal{T}(\kappa) \longrightarrow J(\kappa) \longrightarrow 0,$$

where the right nontrivial map is the obvious one (taking the class of a divisor $D \in \text{Div}^0(Z \setminus \mathcal{T})$ into the class of D modulo $\{\text{div}(f) \mid f \in \mathcal{K}^\times\}$) and the left nontrivial map sends the class of $(x_v)_{v \in \mathcal{T}} \in \bigoplus_{v \in \mathcal{T}} \kappa^\times$ into the class of $\text{div}(f)$, where $f \in \mathcal{K}^\times$, such that $f(v) = x_v$, for all $v \in \mathcal{T}$.

Remark 2.2. Note that since $\tau_{\mathcal{T}}(\kappa)$ and $J(\kappa)$ are divisible groups, $J_{\mathcal{T}}(\kappa)$ is also divisible and we have obvious exact sequences

$$(1) \quad 0 \longrightarrow \tau_{\mathcal{T}}[n] \longrightarrow J_{\mathcal{T}}[n] \longrightarrow J[n] \longrightarrow 0, \quad 0 \longrightarrow T_\ell(\tau_{\mathcal{T}}) \longrightarrow T_\ell(J_{\mathcal{T}}) \longrightarrow T_\ell(J) \longrightarrow 0$$

at the levels of n -torsion points and ℓ -adic Tate modules, for all $n \in \mathbb{N}$ and all primes ℓ .

Clearly, we have a group morphism

$$\text{Div}^0(\mathcal{S}) \xrightarrow{\delta} J_{\mathcal{T}}(\kappa),$$

sending a divisor D into its class modulo $\{\text{div}(f) \mid f \in \mathcal{K}_{\mathcal{T}}^\times\}$. (Recall that $\mathcal{S} \cap \mathcal{T} = \emptyset$.)

Definition 2.3. Deligne's Picard 1-motive $\mathcal{M}_{\mathcal{S}, \mathcal{T}}$ associated to $(Z, \kappa, \mathcal{S}, \mathcal{T})$ is, by definition, the group morphism $\text{Div}^0(\mathcal{S}) \xrightarrow{\delta} J_{\mathcal{T}}(\kappa)$ defined above.

Remark 2.4. One can obviously think of $\mathcal{M}_{\mathcal{S}, \mathcal{T}}$ as associated to $(\mathcal{K}, \kappa, \mathcal{S}, \mathcal{T})$, where κ is an algebraically closed field, \mathcal{K} is a semi-simple finitely generated κ -algebra, of (pure) transcendence degree 1 over κ , and \mathcal{S} and \mathcal{T} are finite, disjoint sets of valuations on \mathcal{K} which are trivial on κ . (In this framework, Z is just a smooth, projective model for \mathcal{K} over κ .)

Next, we recall the construction of the ℓ -adic Tate modules (ℓ -adic realizations) of $\mathcal{M}_{\mathcal{S}, \mathcal{T}}$. For every $n \in \mathbb{N}$, we consider the fiber-product of groups $J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S})$, with respect to the map $\text{Div}^0(\mathcal{S}) \xrightarrow{\delta} J_{\mathcal{T}}(\kappa)$ and the multiplication by n map $J_{\mathcal{T}}(\kappa) \xrightarrow{n} J_{\mathcal{T}}(\kappa)$. An element in this fiber product consists of a pair (\widehat{D}, x) , where $D \in \text{Div}^0(Z \setminus \mathcal{T})$, \widehat{D} is the class of D in $J_{\mathcal{T}}(\kappa)$, $x \in \text{Div}^0(\mathcal{S})$ and $nD - x = \text{div}(f)$, for some $f \in \mathcal{K}_{\mathcal{T}}^\times$. Since $J_{\mathcal{T}}(\kappa)$ is a divisible group, we have a commutative diagram (in the category of abelian groups) whose rows are exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & J_{\mathcal{T}}[n] & \longrightarrow & J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S}) & \longrightarrow & \text{Div}^0(\mathcal{S}) \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \delta \\ 0 & \longrightarrow & J_{\mathcal{T}}[n] & \longrightarrow & J_{\mathcal{T}}(\kappa) & \xrightarrow{n} & J_{\mathcal{T}}(\kappa) \longrightarrow 0 \end{array}$$

In the diagram above, $J_{\mathcal{T}}[n]$ denotes the group of n -torsion points of $J_{\mathcal{T}}$.

Definition 2.5. The group $\mathcal{M}_{\mathcal{S}, \mathcal{T}}[n]$ of n -torsion points of $\mathcal{M}_{\mathcal{S}, \mathcal{T}}$ is defined to be

$$\mathcal{M}_{\mathcal{S}, \mathcal{T}}[n] := (J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S})) \otimes \mathbb{Z}/n\mathbb{Z}.$$

Since $\text{Div}^0(\mathcal{S})$ is a free \mathbb{Z} -module, we have commutative diagrams whose rows are exact

$$(2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & J_{\mathcal{T}}[m] & \longrightarrow & \mathcal{M}_{\mathcal{S}, \mathcal{T}}[m] & \longrightarrow & \text{Div}^0(\mathcal{S}) \otimes \mathbb{Z}/m\mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & J_{\mathcal{T}}[n] & \longrightarrow & \mathcal{M}_{\mathcal{S}, \mathcal{T}}[n] & \longrightarrow & \text{Div}^0(\mathcal{S}) \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0, \end{array}$$

for all $n, m \in \mathbb{N}$ with $n \mid m$, where the left vertical map is multiplication by m/n , the right vertical map is the canonical projection and the middle vertical map is the unique morphism which makes the diagram commute.

Definition 2.6. If ℓ is prime, then the ℓ -adic Tate module $T_\ell(\mathcal{M}_{\mathcal{S}, \mathcal{T}})$ of $\mathcal{M}_{\mathcal{S}, \mathcal{T}}$ is given by

$$T_\ell(\mathcal{M}_{\mathcal{S}, \mathcal{T}}) = \varprojlim_n \mathcal{M}_{\mathcal{S}, \mathcal{T}}[\ell^n],$$

where the projective limit is taken with respect to the surjective maps described in the diagram above.

This way, for every prime ℓ , we obtain exact sequences of free \mathbb{Z}_ℓ -modules

$$0 \longrightarrow T_\ell(J_{\mathcal{T}}) \longrightarrow T_\ell(\mathcal{M}_{\mathcal{S}, \mathcal{T}}) \longrightarrow \text{Div}^0(\mathcal{S}) \otimes \mathbb{Z}_\ell \longrightarrow 0.$$

Remark 2.7. Let us assume that $\text{char}(\kappa) = p$. In this case, if $m \in \mathbb{N}$, then $\tau_{\mathcal{T}}[p^m] = \{1\}$. (There are no non-trivial p -power roots of unity in characteristic p .) Consequently, $J_{\mathcal{T}}[p^m] = J[p^m]$ and $T_p(J_{\mathcal{T}}) = T_p(J)$. Via the exact sequences above, this leads to

$$\mathcal{M}_{\mathcal{S}, \mathcal{T}}[p^m] = \mathcal{M}_{\mathcal{S}, \emptyset}[p^m], \quad T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}) = T_p(\mathcal{M}_{\mathcal{S}, \emptyset}).$$

Remark 2.8. Obviously, for all $n, m \in \mathbb{N}$, such that $n \mid m$, we have injective group morphisms

$$\mathcal{M}_{\mathcal{S}, \mathcal{T}}[n] \longrightarrow \mathcal{M}_{\mathcal{S}, \mathcal{T}}[m],$$

given by $(\widehat{D}, x) \otimes \widehat{1} \longrightarrow (\widehat{D}, \frac{m}{n} \cdot x) \otimes \widehat{1}$. Therefore, one can define $\mathcal{M}_{\mathcal{S}, \mathcal{T}}[\ell^\infty] := \varinjlim_m \mathcal{M}_{\mathcal{S}, \mathcal{T}}[\ell^m]$, for any prime ℓ , where the injective limit is taken with respect to the morphisms constructed above. It is easy to see that $\mathcal{M}_{\mathcal{S}, \mathcal{T}}[\ell^\infty]$ is a divisible \mathbb{Z}_ℓ -module of finite co-rank and there is a canonical \mathbb{Z}_ℓ -module isomorphism

$$T_\ell(\mathcal{M}_{\mathcal{S}, \mathcal{T}}) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathcal{M}_{\mathcal{S}, \mathcal{T}}[\ell^\infty]).$$

Proposition 2.9. For every $n \in \mathbb{N}$, we have a canonical group isomorphism

$$\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}/\mathcal{K}_{\mathcal{T}}^{\times n} \xrightarrow{\sim} \mathcal{M}_{\mathcal{S}, \mathcal{T}}[n],$$

where the notations are as in Definition 2.1.

Proof. Let us fix an $n \in \mathbb{N}$. An element in the fiber product $J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S})$ consists of a pair (\widehat{D}, x) , where $D \in \text{Div}^0(Z \setminus \mathcal{T})$, \widehat{D} is the class of D in $J_{\mathcal{T}}(\kappa)$, $x \in \text{Div}^0(\mathcal{S})$ and $nD - x = \text{div}(f)$, for some $f \in \mathcal{K}_{\mathcal{T}}^\times$. First, we define a group morphism

$$J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S}) \xrightarrow{\phi} \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}/\mathcal{K}_{\mathcal{T}}^{\times n}$$

given by $\phi(\widehat{D}, x) = \widehat{f}$, where $f \in \mathcal{K}_{\mathcal{T}}^\times$ such that $\text{div}(f) = nD - x$, and \widehat{f} is the class of f in $\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}/\mathcal{K}_{\mathcal{T}}^{\times n}$. We claim that ϕ is a well-defined, surjective group morphism. We check this next.

Step 1. ϕ is a well-defined function. First, let $(\widehat{D}, x) = (\widehat{D}', x)$ in $J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S})$. The definitions show that this means that

$$D - D' = \text{div}(g), \text{ for some } g \in \mathcal{K}_{\mathcal{T}}^\times$$

$$nD - x = \text{div}(f) \text{ and } nD' - x = \text{div}(f'), \text{ for some } f, f' \in \mathcal{K}_{\mathcal{T}}^\times.$$

When combined, these equalities imply that $\text{div}(ff'^{-1}) = \text{div}(g^n)$. Now, one has to work on each connected component of Z separately. So, without loss of generality, we may assume that Z is connected. If $\mathcal{T} \neq \emptyset$, the last equality implies that $ff'^{-1} = g^n$. If $\mathcal{T} = \emptyset$, then we have $\mathcal{K}_{\mathcal{T}}^\times = \mathcal{K}^\times$ and $ff'^{-1} = g^n \cdot \alpha$, for some constant rational function α on Z . However, since κ is algebraically closed, α is an n -power in \mathcal{K}^\times and therefore $ff'^{-1} = g'^n$, for some $g' \in \mathcal{K}_{\mathcal{T}}^\times$. The conclusion is that in all cases we have $\widehat{f} = \widehat{f}'$ in $\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}/\mathcal{K}_{\mathcal{T}}^{\times n}$, which concludes Step 1.

Finally, note that if (\widehat{D}, x) and f are as above, then \widehat{f} is uniquely determined (despite the fact that f itself is not.) Indeed, the above argument shows that for $\mathcal{T} \neq \emptyset$ this is obvious, while for $\mathcal{T} = \emptyset$ this follows from the fact that κ is algebraically closed

Step 2. ϕ is a surjective group morphism. Only the surjectivity of ϕ requires a proof. This is an immediate consequence of the equality

$$\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)} = \{f \in \mathcal{K}_{\mathcal{T}}^\times \mid \text{div}(f) = nD - x, D \in \text{Div}^0(Z \setminus \mathcal{T}), x \in \text{Div}^0(\mathcal{S})\}.$$

The inclusion of the right-hand side into the left-hand side is obvious from the definition of $\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}$. Now, let $f \in \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}$ and let $\text{div}(f) = nD + y$, where $D \in \text{Div}(Z)$ and $y \in \text{Div}(\mathcal{S})$. If y is the 0 divisor, then we obviously have $D \in \text{Div}^0(Z \setminus \mathcal{T})$ and we are done. (Keep in mind that $\deg(\text{div}(f)) = 0$!) If y is not the 0 divisor, let v be a closed point in the support of y . Then, we have

$$\text{div}(f) = n(D - \deg(D)v) - (-n \cdot \deg(D)v - y).$$

It is immediate that

$$D - \deg(D)v \in \text{Div}^0(Z \setminus \mathcal{T}) \text{ and } (-n \cdot \deg(D)v - y) \in \text{Div}^0(\mathcal{S}),$$

which concludes Step 2. Therefore ϕ is a surjective group morphism.

Step 3. Now, we claim that

$$\ker(\phi) = n(J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S})).$$

Indeed, let $(\widehat{D}, x) \in J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S})$, such that $\phi(\widehat{D}, x) = \widehat{0}$. This is equivalent to

$$nD - x = \text{div}(g^n) = n \cdot \text{div}(g),$$

for some $g \in \mathcal{K}_{\mathcal{T}}^\times$. This implies that $x = nx'$, for some $x' \in \text{Div}^0(\mathcal{S})$. However, $J_{\mathcal{T}}(\kappa)$ is divisible, so there exists $\widehat{D}' \in J_{\mathcal{T}}(\kappa)$, such that $n\widehat{D}' = \widehat{D}$. This means that $nD' - D = \text{div}(f')$, for some $f' \in \mathcal{K}_{\mathcal{T}}^\times$. When combined, these equalities lead to

$$(\widehat{D}, x) = n(\widehat{D}', x'), \quad nD' - x' = \text{div}(gf'),$$

which proves that $\ker(\phi) \subseteq n(J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S}))$. The opposite inclusion is obvious.

Now, Steps 1–3 combined with the definition of $\mathcal{M}_{\mathcal{S}, \mathcal{T}}[n]$ lead to the conclusion that the group morphism ϕ constructed above factors through a group isomorphism $\tilde{\phi}$

$$\begin{array}{ccc} J_{\mathcal{T}}(\kappa) \times_{J_{\mathcal{T}}(\kappa)}^n \text{Div}^0(\mathcal{S}) & \xrightarrow{\phi} & \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)} / \mathcal{K}_{\mathcal{T}}^{\times n} \\ \downarrow & \nearrow \tilde{\phi} \sim & \\ \mathcal{M}_{\mathcal{S}, \mathcal{T}}[n] & & \end{array}$$

This concludes the proof of the Proposition. \square

Remark 2.10. Assume that κ_0 is a subfield of κ and \mathcal{A} is a group of automorphisms of Z in the category of κ_0 -schemes. Further, assume that the sets of closed points \mathcal{S} and \mathcal{T} are (set-wise, not necessarily point-wise) invariant under the action of \mathcal{A} . Then it is easy to see that literally all the groups associated to the set of data $(Z, \kappa, \mathcal{S}, \mathcal{T})$ in the current section are endowed with a

natural \mathcal{A} -action (and can be viewed as $\mathbb{Z}[\mathcal{A}]$ -modules). Most importantly, all the group morphisms considered above become morphisms of $\mathbb{Z}[\mathcal{A}]$ -modules. In particular, the isomorphism

$$\mathcal{K}_{S,\mathcal{T}}^{(n)}/\mathcal{K}_{\mathcal{T}}^{\times n} \xrightarrow{\sim} \mathcal{M}_{S,\mathcal{T}}[n]$$

in the statement of Proposition 2.9 is an isomorphism of $\mathbb{Z}[\mathcal{A}]$ -modules. A typical example is the following. Assume that κ_0 is a subfield of κ , such that the field extension κ/κ_0 is Galois of (profinite) Galois group \mathcal{G} . Assume that $Z_0 \rightarrow Z'_0$ is a (finite) Galois cover of smooth, projective curves over κ_0 , of (finite) Galois group G . Let S'_0 and T'_0 be two finite, disjoint sets of closed points on Z'_0 . Let $Z := Z_0 \times_{\kappa_0} \kappa$ and $Z' := Z'_0 \times_{\kappa_0} \kappa$ be the usual extensions of scalars to κ . Further, let \mathcal{S} and \mathcal{T} be the sets of closed points on Z sitting above points in S'_0 and T'_0 , respectively. Then the (profinite) group $\mathcal{A} := G \times \mathcal{G}$ is a group of κ_0 -scheme automorphisms of Z which fixes the sets of closed points \mathcal{S} and \mathcal{T} (set-wise.) In this case, it is important to note that if one views the groups associated above to $(Z, \kappa, \mathcal{S}, \mathcal{T})$ as topological groups endowed with discrete topologies, then the action of \mathcal{G} (viewed as a topological group with its profinite topology) on these groups is continuous. This implies directly that $T_\ell(\mathcal{M}_{S,\mathcal{T}})$ (endowed with the ℓ -adic topology now) is a topological $\mathbb{Z}_\ell[G][[\mathcal{G}]]$ -module, for all primes ℓ .

3. Cohomological triviality of ℓ -adic realizations of Picard 1-motives

In what follows, $Z \rightarrow Z'$ will denote a (finite) G -Galois cover of connected, smooth, projective curves over an algebraically closed field κ , and \mathcal{S} and \mathcal{T} are G -invariant, finite, non-empty, disjoint sets of closed points on Z , such that \mathcal{S} contains the ramified locus of the cover. Let \mathcal{S}' and \mathcal{T}' denote the sets of closed points on Z' , sitting below points in \mathcal{S} and \mathcal{T} , respectively. We denote by \mathcal{M} and \mathcal{M}' the Picard 1-motives associated to the sets of data $(Z, \kappa, \mathcal{S}, \mathcal{T})$ and $(Z', \kappa, \mathcal{S}', \mathcal{T}')$, respectively. As usual, \mathcal{K} and \mathcal{K}' denote the κ -algebras of rational functions on Z and Z' , respectively.

Note that we have natural inclusions $\mathcal{K}_{S',\mathcal{T}'}^{(n)} \subseteq \mathcal{K}_{S,\mathcal{T}}^{(n)}$ and $\mathcal{K}_{\mathcal{T}'}^{\times n} \subseteq \mathcal{K}_{\mathcal{T}}^{\times n}$. Now, since \mathcal{K}/\mathcal{K}' is unramified at points in \mathcal{T}' , we have an equality $\mathcal{K}_{\mathcal{T}}^{\times} \cap \mathcal{K}' = \mathcal{K}_{\mathcal{T}'}^{\times}$. Since $\mathcal{T} \neq \emptyset$, the multiplicative group $\mathcal{K}_{\mathcal{T}}^{\times}$ has no torsion. This implies right away that $\mathcal{K}_{\mathcal{T}'}^{\times n} \cap \mathcal{K}'^{\times} = \mathcal{K}_{\mathcal{T}'}^{\times n}$. (Indeed, simply take G -invariants in the $\mathbb{Z}[G]$ -module isomorphism $\mathcal{K}_{\mathcal{T}}^{\times} \xrightarrow{\sim} \mathcal{K}_{\mathcal{T}'}^{\times n}$ given by $x \rightarrow x^n$ and keep in mind that the extension \mathcal{K}/\mathcal{K}' is G -Galois.) These observations combined with Proposition 2.9 lead to natural injective G -module morphisms

$$\mathcal{M}'[n] \hookrightarrow \mathcal{M}[n], \quad T_\ell(\mathcal{M}') \hookrightarrow T_\ell(\mathcal{M}),$$

for all natural numbers n and all prime numbers ℓ , where G acts trivially on $\mathcal{M}'[n]$ and $T_\ell(\mathcal{M}')$.

Theorem 3.1. *With notations as above, we have*

$$\mathcal{M}[n]^G = \mathcal{M}'[n], \quad T_\ell(\mathcal{M})^G = T_\ell(\mathcal{M}')$$

for all natural numbers n and all prime numbers ℓ .

Proof. Obviously, the second equality above is a consequence of the first (by passing to a projective limit.) Now, let us fix a natural number n . According to Proposition 2.9, we have an exact sequence of $\mathbb{Z}[G]$ -modules

$$0 \longrightarrow \mathcal{K}_{\mathcal{T}}^{\times n} \longrightarrow \mathcal{K}_{S,\mathcal{T}}^{(n)} \longrightarrow \mathcal{M}[n] \longrightarrow 0.$$

If we apply G –cohomology to the exact sequence above, we obtain an exact sequence

$$0 \longrightarrow (\mathcal{K}_{\mathcal{T}}^{\times n})^G \longrightarrow \left(\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}\right)^G \longrightarrow \mathcal{M}[n]^G \longrightarrow \widehat{H}^1(G, \mathcal{K}_{\mathcal{T}}^{\times n}) .$$

This shows that the first equality in the Theorem follows if we show that

$$(3) \quad (\mathcal{K}_{\mathcal{T}}^{\times n})^G = \mathcal{K}_{\mathcal{T}'}^{\times n}, \quad \left(\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}\right)^G = \mathcal{K}_{\mathcal{S}', \mathcal{T}'}^{\times n}, \quad \widehat{H}^1(G, \mathcal{K}_{\mathcal{T}}^{\times n}) = 0 .$$

Step 1. The first equality above is equivalent to $\mathcal{K}_{\mathcal{T}}^{\times n} \cap \mathcal{K}'^{\times} = \mathcal{K}_{\mathcal{T}'}^{\times n}$, which was proved above.

Step 2. The third equality is proved as follows. First, we observe that since $\mathcal{K}_{\mathcal{T}}^{\times}$ has no torsion, we have an isomorphism of $\mathbb{Z}[G]$ –modules $\mathcal{K}_{\mathcal{T}}^{\times} \xrightarrow{\sim} \mathcal{K}_{\mathcal{T}}^{\times n}$ given by the n –power map. Now, we let $\mathcal{K}_{(\mathcal{T})}^{\times} := \{f \in \mathcal{K} \mid f(w) \neq 0, \text{ for all } w \in \mathcal{T}\}$. We have exact sequences of $\mathbb{Z}[G]$ –modules

$$0 \longrightarrow \mathcal{K}_{\mathcal{T}}^{\times} \longrightarrow \mathcal{K}_{(\mathcal{T})}^{\times} \xrightarrow{\text{ev}_{\mathcal{T}}} \bigoplus_{w \in \mathcal{T}} \kappa(w)^{\times} \longrightarrow 0, \quad 0 \longrightarrow \mathcal{K}_{(\mathcal{T})}^{\times} \longrightarrow \mathcal{K}^{\times} \xrightarrow{\text{div}_{\mathcal{T}}} \text{Div}(\mathcal{T}) \longrightarrow 0,$$

where $\kappa(w)$ is the residue field associated to w (in fact isomorphic to κ), the map $\text{ev}_{\mathcal{T}}$ is the \mathcal{T} –evaluation map taking $f \in \mathcal{K}_{(\mathcal{T})}^{\times}$ into $(f(w))_{w \in \mathcal{T}}$ and $\text{div}_{\mathcal{T}}(f)$ is the piece of the divisor of f supported on \mathcal{T} , for all $f \in \mathcal{K}^{\times}$. Note that the maps $\text{ev}_{\mathcal{T}}$ and $\text{div}_{\mathcal{T}}$ are surjective as a consequence of the weak approximation theorem applied to the (independent) valuations of \mathcal{K} associated to the points in \mathcal{T} . Now, since \mathcal{K}/\mathcal{K}' is unramified at points on \mathcal{T}' and κ is algebraically closed, we have isomorphisms of $\mathbb{Z}[G]$ –modules

$$\bigoplus_{w \in \mathcal{T}} \kappa(w)^{\times} \xrightarrow{\sim} \mathbb{Z}[G] \otimes_{\mathbb{Z}} \bigoplus_{v \in \mathcal{T}'} \kappa(v)^{\times}, \quad \text{Div}(\mathcal{T}) \xrightarrow{\sim} \mathbb{Z}[G] \otimes_{\mathbb{Z}} \text{Div}(\mathcal{T}'),$$

where G acts trivially on $\kappa(v)^{\times}$ and $\text{Div}(\mathcal{T}')$. This shows that $\bigoplus_{w \in \mathcal{T}} \kappa(w)^{\times}$ and $\text{Div}(\mathcal{T})$ are induced G –modules and therefore G –cohomologically trivial. Consequently, we have group–isomorphisms

$$\widehat{H}^i(G, \mathcal{K}_{\mathcal{T}}^{\times}) \xrightarrow{\sim} \widehat{H}^i(G, \mathcal{K}_{(\mathcal{T})}^{\times}) \xrightarrow{\sim} \widehat{H}^i(G, \mathcal{K}^{\times}),$$

for all $i \in \mathbb{Z}$. In particular, Hilbert's Theorem 90 leads to

$$\widehat{H}^1(G, \mathcal{K}_{\mathcal{T}}^{\times n}) \xrightarrow{\sim} \widehat{H}^1(G, \mathcal{K}_{(\mathcal{T})}^{\times}) \xrightarrow{\sim} \widehat{H}^1(G, \mathcal{K}^{\times}) = 0,$$

which proves the third equality in (3) above.

Step 3. Now, we proceed to proving the second equality in (3). By the definition of $\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}$, we have the following exact sequence of G –modules

$$0 \longrightarrow \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)} \longrightarrow \mathcal{K}_{\mathcal{T}}^{\times} \xrightarrow{\delta_{\mathcal{S}, n}} \text{Div}(Z \setminus \mathcal{S}) \otimes \mathbb{Z}/n\mathbb{Z},$$

where $\delta_{\mathcal{S}, n}(f)$ is the piece of the divisor of f supported away from \mathcal{S} modulo n , for all $f \in \mathcal{K}_{\mathcal{T}}^{\times}$. However, since \mathcal{K}/\mathcal{K}' is unramified away from \mathcal{S} and κ is algebraically closed, we have an isomorphism of G –modules

$$\text{Div}(Z \setminus \mathcal{S}) \xrightarrow{\sim} \text{Div}(Z' \setminus \mathcal{S}') \otimes_{\mathbb{Z}} \mathbb{Z}[G].$$

Therefore, after taking G –invariants in the exact sequence above, we obtain an exact sequence

$$0 \longrightarrow \left(\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}\right)^G \longrightarrow \mathcal{K}_{\mathcal{T}'}^{\times} \xrightarrow{\delta_{\mathcal{S}', n}} \text{Div}(Z' \setminus \mathcal{S}') \otimes \mathbb{Z}/n\mathbb{Z},$$

at the level of Z' . This shows that $\left(\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}\right)^G = \mathcal{K}'_{\mathcal{S}', \mathcal{T}'}$, as required. \square

Definition 3.2. We let g_Z denote the genus of Z . If $\text{char}(\kappa) = p$, we let γ_Z denote the Hasse-Witt invariant of Z .

Remark 3.3. Recall that if J_Z is the Jacobian of Z , we have

$$2g_Z = \text{rank}_{\mathbb{Z}_\ell} T_\ell(J_Z) = \dim_{\mathbb{F}_\ell} J_Z[\ell],$$

for all prime numbers $\ell \neq \text{char}(\kappa)$. Also, if $\text{char}(\kappa) = p > 0$, recall that

$$\gamma_Z = \text{rank}_{\mathbb{Z}_p} T_p(J_Z) = \dim_{\mathbb{F}_p} J_Z[p], \quad g_Z \geq \gamma_Z.$$

The reader who is unfamiliar with the more standard definitions of g_Z and γ_Z should feel free to take the above equalities as definitions.

The following well-known theorem captures the behavior of the invariants g and γ in Galois covers.

Theorem 3.4. Assume that $Z \rightarrow Z'$ is a (finite) G -Galois cover of smooth, projective, connected curves over an algebraically closed field κ and let \mathcal{S} be a G -invariant set of closed points on Z , containing the ramification locus of the cover. The following hold.

(1) **(The Hurwitz genus formula.)** If $|G|$ is coprime to $\text{char}(\kappa)$, then

$$(2g_Z - 2) = |G|(2g_{Z'} - 2) + \sum_{w \in \mathcal{S}} (e_w - 1),$$

where e_w denotes the ramification index of w in the cover $Z \rightarrow Z'$.

(2) **(The Deuring-Shafarevich formula.)** If $\text{char}(\kappa) = p > 0$ and G is a p -group, then

$$(\gamma_Z - 1) = |G|(\gamma_{Z'} - 1) + \sum_{w \in \mathcal{S}} (e_w - 1).$$

Proof. See Corollary 2.4, in Chpt. IV of [9] for part (1) and formula (1.1) in [16] and Theorem 4.2 in [19] for part (2). \square

Remark 3.5. With notations as above, since κ is algebraically closed (which kills inertia, meaning that the residue field extensions associated to closed points on Z' in the cover $Z \rightarrow Z'$ are all trivial) and \mathcal{K}/\mathcal{K}' is Galois, we have an equality

$$\sum_{w \in \mathcal{S}} e_w = |G| \cdot |\mathcal{S}'|,$$

where \mathcal{S}' is the set of closed points on Z' sitting below those in \mathcal{S} . This leads to the following equivalent formulation of the equalities in the theorem above.

$$(2g_Z - 2 + |\mathcal{S}|) = |G|(2g_{Z'} - 2 + |\mathcal{S}'|), \quad (\gamma_Z - 1 + |\mathcal{S}|) = |G|(\gamma_{Z'} - 1 + |\mathcal{S}'|).$$

Theorem 3.6. With notations as above, let ℓ be a prime number and assume that G is an ℓ -group. Then, $\mathcal{M}[\ell]$ is a free $\mathbb{F}_\ell[G]$ -module of rank $r_{\mathcal{M}', \ell}$, where

$$r_{\mathcal{M}', \ell} := \begin{cases} (2g_{Z'} - 2 + |\mathcal{S}'| + |\mathcal{T}'|), & \ell \neq \text{char}(\kappa); \\ (\gamma_{Z'} - 1 + |\mathcal{S}'|), & \ell = \text{char}(\kappa). \end{cases}$$

Proof. The main ingredients needed in the proof are Theorem 3.1 above and the following result.

Proposition 3.7. *Let ℓ be a prime number, k a field of characteristic ℓ , G a finite ℓ -group and M a finitely generated $k[G]$ -module. If*

$$\dim_k M \geq |G| \cdot \dim_k M^G,$$

then M is a free $k[G]$ -module of rank $r := \dim_k M^G$.

Proof. See [16], Proposition 2, §4. \square

Now, the idea is to apply this proposition to $M := \mathcal{M}[\ell]$ and $k := \mathbb{F}_\ell$. The exact sequences (1) and (2) for $n := \ell$ give the following equality.

$$\dim_{\mathbb{F}_\ell} \mathcal{M}[\ell] = \dim_{\mathbb{F}_\ell} J_Z[\ell] + \dim_{\mathbb{F}_\ell} \tau_{\mathcal{T}}[\ell] + \dim_{\mathbb{F}_\ell} \text{Div}^0(\mathcal{S}) \otimes \mathbb{F}_\ell.$$

However, from the definitions we have

$$\dim_{\mathbb{F}_\ell} \text{Div}^0(\mathcal{S}) \otimes \mathbb{F}_\ell = |\mathcal{S}| - 1, \quad \dim_{\mathbb{F}_\ell} \tau_{\mathcal{T}}[\ell] = \begin{cases} |\mathcal{T}| - 1, & \ell \neq \text{char}(\kappa); \\ 0, & \ell = \text{char}(\kappa). \end{cases}$$

Now, we combine these equalities with Remark 3.3 above, to conclude that

$$\dim_{\mathbb{F}_\ell} \mathcal{M}[\ell] = r_{\mathcal{M}, \ell},$$

for all prime numbers ℓ . Obviously, the same equality holds for the 1-motive \mathcal{M}' . Next, note that since the cover $Z \rightarrow Z'$ is unramified at points in \mathcal{T}' and κ is algebraically closed, we have

$$|\mathcal{T}| = |G| \cdot |\mathcal{T}'|.$$

Next, we combine the last two equalities with Remark 3.5 and Theorem 3.1 to obtain

$$\dim_{\mathbb{F}_\ell} \mathcal{M}[\ell] = |G| \cdot \dim_{\mathbb{F}_\ell} \mathcal{M}'[\ell] = |G| \cdot \dim_{\mathbb{F}_\ell} \mathcal{M}[\ell]^G.$$

Finally, as promised, we apply Proposition 3.7 to $M := \mathcal{M}[\ell]$ and $k := \mathbb{F}_\ell$ to conclude that $\mathcal{M}[\ell]$ is a free $\mathbb{F}_\ell[G]$ -module of rank $r_{\mathcal{M}', \ell}$. \square

Now, we are ready to pass from the study of $\mathcal{M}[\ell]$ to that of $T_\ell(\mathcal{M})$. The following remark makes this possible.

Remark 3.8. *Let ℓ be a prime number. Since $\mathcal{M}[\ell^\infty] := \varinjlim_m \mathcal{M}[\ell^m]$ is a divisible \mathbb{Z}_ℓ -module (see Remark 2.8) and $T_\ell(\mathcal{M})$ is \mathbb{Z}_ℓ -free, we have a canonical exact sequence of $\mathbb{Z}_\ell[G]$ -modules*

$$(4) \quad 0 \longrightarrow T_\ell(\mathcal{M}) \xrightarrow{\times \ell} T_\ell(\mathcal{M}) \longrightarrow \mathcal{M}[\ell] \longrightarrow 0.$$

Indeed, if we apply the (exact) functor $ \rightarrow \text{Hom}_{\mathbb{Z}_\ell}(*, \mathcal{M}[\ell^\infty])$ to the exact sequence*

$$0 \longrightarrow \mathbb{Z}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow 0$$

and make the canonical identification $T_\ell(\mathcal{M}) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathcal{M}[\ell^\infty])$, we obtain the following exact sequence

$$0 \longrightarrow T_\ell(\mathcal{M}) \longrightarrow \text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Q}_\ell, \mathcal{M}[\ell^\infty]) \longrightarrow \mathcal{M}[\ell^\infty] \longrightarrow 0.$$

Now, exact sequence (4) is obtained by applying the snake lemma to the multiplication by ℓ -endomorphism of the exact sequence above, and remarking that multiplication by ℓ is an automorphism of the \mathbb{Q}_ℓ -vector space $\text{Hom}_{\mathbb{Z}_\ell}(\mathbb{Q}_\ell, \mathcal{M}[\ell^\infty])$.

Theorem 3.9. *As above, let $\pi : Z \rightarrow Z'$ be a (finite) G -Galois cover of smooth, projective, connected curves defined over an algebraically closed field κ . Let \mathcal{S} and \mathcal{T} denote two finite, non-empty G -invariant sets of closed points on Z , such that \mathcal{S} contains the ramification locus \mathcal{S}_{ram} of the cover. Let \mathcal{M} and \mathcal{M}' be the Picard 1-motives associated to the data $(Z, \kappa, \mathcal{S}, \mathcal{T})$ and $(Z', \kappa, \mathcal{S}' := \pi(\mathcal{S}), \mathcal{T}' := \pi(\mathcal{T}))$, respectively. Let ℓ be a prime. Then*

- (1) $T_\ell(\mathcal{M})$ is a projective $\mathbb{Z}_\ell[G]$ -module of finite rank;
- (2) If G is an ℓ -group, then $T_\ell(\mathcal{M})$ is a free $\mathbb{Z}_\ell[G]$ -module of rank $r_{\mathcal{M}', \ell}$.

Proof. Let ℓ be a fixed prime number. Theorem 3.6 implies that $\mathcal{M}[\ell]$ is G -cohomologically trivial. Indeed, since $\mathcal{M}[\ell]$ is an ℓ -group, it suffices to show that $\mathcal{M}[\ell]$ is $G^{(\ell)}$ -cohomologically trivial, where $G^{(\ell)}$ is an ℓ -Sylow subgroup of G . However, Theorem 3.6 applied to the $G^{(\ell)}$ -cover $Z \rightarrow Z^{(\ell)}$, where $Z^{(\ell)}$ is a smooth projective model associated to the field $\mathcal{K}^{G^{(\ell)}}$ (maximal subfield of \mathcal{K} fixed by $G^{(\ell)}$), implies that $\mathcal{M}[\ell]$ is $\mathbb{F}_\ell[G^{(\ell)}]$ -free and therefore $G^{(\ell)}$ -cohomologically trivial. Now, the long exact sequence in G' -cohomology associated to the short exact sequence (4) for an arbitrary subgroup G' of G implies that the multiplication by ℓ maps give group isomorphisms

$$\widehat{H}^i(G', T_\ell(\mathcal{M})) \xrightarrow{\times \ell} \widehat{H}^i(G', T_\ell(\mathcal{M})) ,$$

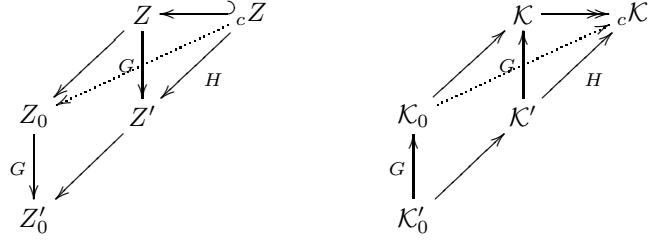
for all $i \in \mathbb{Z}$. However, since the cohomology groups $\widehat{H}^i(G', T_\ell(\mathcal{M}))$ are finitely generated (torsion) \mathbb{Z}_ℓ -modules, this implies that $\widehat{H}^i(G', T_\ell(\mathcal{M})) = 0$, for all i and G' as above. This shows that $T_\ell(\mathcal{M})$ is G -cohom. trivial. Since it is \mathbb{Z}_ℓ -free, $T_\ell(\mathcal{M})$ is $\mathbb{Z}_\ell[G]$ -projective, which proves part (1).

Next, assume that G is an ℓ -group. Then the ring $\mathbb{Z}_\ell[G]$ is local. Consequently, the finitely generated projective $\mathbb{Z}_\ell[G]$ -module $T_\ell(\mathcal{M})$ is $\mathbb{Z}_\ell[G]$ -free (see Lemma 1.2 of [15]) of rank r , say. However, the exact sequence (4) gives an $\mathbb{F}_\ell[G]$ -module isomorphism

$$T_\ell(\mathcal{M}) \otimes_{\mathbb{Z}_\ell[G]} \mathbb{F}_\ell[G] \simeq \mathcal{M}[\ell] .$$

Consequently, Theorem 3.6 implies that $r = r_{\mathcal{M}', \ell}$, concluding the proof of part (2). \square

Next, we extend the theorem above to a situation of particular arithmetic interest to us, in which the curve Z is not necessarily connected. For that purpose, we let κ_0 denote a perfect field of arbitrary characteristic (e.g. $\kappa_0 := \mathbb{F}_q$, with q a power of a prime p) and we let $Z_0 \rightarrow Z'_0$ denote a G -Galois cover of smooth, projective curves over κ_0 . We assume that Z'_0 is geometrically connected, but Z_0 may not satisfy this property. More explicitly, if we let κ denote the algebraic closure of κ_0 , then the first of the smooth projective κ -curves $Z' := Z'_0 \times_{\kappa_0} \kappa$ and $Z := Z_0 \times_{\kappa_0} \kappa$ is connected, but the second may not be. This is equivalent to saying that κ_0 is algebraically closed in the function field $\mathcal{K}'_0 := \kappa_0(Z'_0)$, but may not be so inside $\mathcal{K}_0 := \kappa_0(Z_0)$. Let $\mathcal{K}' := \mathcal{K}'_0 \otimes_{\kappa_0} \kappa$ be the field of rational functions on Z' and $\mathcal{K} := \mathcal{K}_0 \otimes_{\kappa_0} \kappa$ the ring of rational functions on Z . Then $Z \rightarrow Z'$ is a G -Galois cover of smooth projective curves over κ and \mathcal{K} is a G -Galois \mathcal{K}' -algebra. In fact, if we let $H := \text{Gal}(\mathcal{K}_0/\mathcal{K}_0 \cap \mathcal{K}')$, then Z has $[G : H]$ distinct (κ -isomorphic) connected components and \mathcal{K} is a direct sum of as many fields which are mutually isomorphic as \mathcal{K}' -algebras (isomorphic to a field compositum ${}^c\mathcal{K} := \mathcal{K}_0 \cdot \kappa$.) If we fix such a compositum ${}^c\mathcal{K}$ (viewed as a quotient of \mathcal{K} by one of its maximal ideals), then ${}^c\mathcal{K}$ is an H -Galois extension of \mathcal{K}' and it is the function field of a smooth, projective, connected κ -curve cZ which is an H -Galois cover of Z' . (Obviously, the subscript “ c ” above stands for “connected”.)



Theorem 3.10. *Let $\pi_0 : Z_0 \rightarrow Z'_0$ be a (finite) G -cover of smooth, projective curves defined over a perfect field κ_0 . Let κ denote the algebraic closure of κ_0 , let $Z := Z_0 \times_{\kappa_0} \kappa$ and $Z' := Z'_0 \times_{\kappa_0} \kappa$ and $\pi := \pi_0 \times \mathbf{1}_\kappa$. Assume that Z' is connected. Let S'_0 and T'_0 be finite, disjoint sets of closed points on Z'_0 , such that S'_0 is non-empty and contains the ramified locus of the cover $Z_0 \rightarrow Z'_0$. Let \mathcal{S} and \mathcal{T} be the sets of closed points on Z sitting above points in S'_0 and T'_0 , respectively. Let \mathcal{M} and \mathcal{M}' denote the Picard 1-motives associated to the sets of data $(Z, \kappa, \mathcal{S}, \mathcal{T})$ and $(Z', \kappa, \mathcal{S}' := \pi(\mathcal{S}), \mathcal{T}' := \pi(\mathcal{T}))$, respectively. Let ℓ be a prime. Then*

- (1) $T_\ell(\mathcal{M})$ is a projective $\mathbb{Z}_\ell[G]$ -module of finite rank;
- (2) If G is an ℓ -group, then $T_\ell(\mathcal{M})$ is a free $\mathbb{Z}_\ell[G]$ -module of rank $r_{\mathcal{M}', \ell}$.

Proof. Recall that $\mathcal{K} \xrightarrow{\sim} \mathcal{K}_0 \otimes_{\mathcal{K}'_0} \mathcal{K}'$ is a (finite) direct sum of fields which are mutually isomorphic as $\mathcal{K}'[H]$ -algebras. The $\mathcal{K}'[H]$ -algebra morphism $\pi : \mathcal{K} \rightarrow {}_c \mathcal{K}$ induces a $\mathcal{K}'[H]$ -algebra isomorphism between exactly one of these direct summands and ${}_c \mathcal{K}$. The inverse of this isomorphism produces a $\mathbb{Z}[H]$ -linear section $\iota : {}_c \mathcal{K}^\times \hookrightarrow \mathcal{K}^\times$ of the $\mathbb{Z}[H]$ -module morphism $\mathcal{K}^\times \rightarrow {}_c \mathcal{K}^\times$ induced by π . It is easy to check that this gives an isomorphism of $\mathbb{Z}[G]$ -modules

$$\iota \otimes \mathbf{1} : {}_c \mathcal{K}^\times \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \xrightarrow{\sim} \mathcal{K}^\times.$$

Now, let \mathcal{S}_c and \mathcal{T}_c denote the sets of closed points on ${}_c Z$, sitting above points in S'_0 and T'_0 , respectively. Then $\iota \otimes \mathbf{1}$ induces isomorphisms of $\mathbb{Z}[G]$ -modules

$$\iota \otimes \mathbf{1} : {}_c \mathcal{K}_{\mathcal{S}_c, \mathcal{T}_c}^{(n)} \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \xrightarrow{\sim} \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(n)}, \quad \iota \otimes \mathbf{1} : {}_c \mathcal{K}_{\mathcal{T}_c}^{\times n} \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \xrightarrow{\sim} \mathcal{K}_{\mathcal{T}}^{\times n},$$

for all $n \in \mathbb{N}$ (se Definition 2.1.) Consequently, if we denote by \mathcal{M}_c the Picard 1-motive associated to data $({}_c Z, \kappa, \mathcal{S}_c, \mathcal{T}_c)$, then we have isomorphisms

$$\mathcal{M}_c[n] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \xrightarrow{\sim} \mathcal{M}[n], \quad T_\ell(\mathcal{M}_c) \otimes_{\mathbb{Z}_\ell[H]} \mathbb{Z}_\ell[G] \xrightarrow{\sim} T_\ell(\mathcal{M}),$$

of $\mathbb{Z}/n\mathbb{Z}[G]$ and $\mathbb{Z}_\ell[G]$ -modules, respectively, for all $n \in \mathbb{N}$ and all prime numbers ℓ . Now, since Z_c is connected, Theorem 3.9 implies that $T_\ell(\mathcal{M}_c)$ is $\mathbb{Z}_\ell[H]$ -projective of finite rank and free of rank $r_{\mathcal{M}', \ell}$, if G is an ℓ -group. Both parts (1) and (2) of our Theorem are immediate consequences of these facts combined with the second isomorphism above. \square

4. Fitting ideals of ℓ -adic realizations of Picard 1-motives defined over finite fields

In this section, we apply Theorem 3.10 above in the arithmetically interesting situation where κ_0 is a finite field ($\kappa_0 = \mathbb{F}_q$, where q is a power of a prime p) and G is abelian. In this case, $\mathcal{K}_0/\mathcal{K}'_0$ is an abelian G -Galois extension of characteristic p global fields (see the set-up and notations leading into Theorem 3.10.) Our goal is to compute the first Fitting ideal of $T_\ell(\mathcal{M})$ over a certain profinite ℓ -adic group ring in terms of special values of G -equivariant L -functions, for all prime numbers ℓ .

4.1. Algebraic preliminaries. Let R be a commutative ring with 1 , P a finitely generated, projective R -module and $f \in \text{End}_R(P)$. Then, the determinant $\det_R(f \mid P)$ of f acting on P makes sense and it is defined as follows. We take a finitely generated R -module Q , such that $P \oplus Q$ is a (finitely generated) free R -module, then we let $f \oplus \mathbf{1}_Q \in \text{End}_R(P \oplus Q)$, where $\mathbf{1}_Q$ is the identity of Q , and define

$$\det_R(f \mid P) := \det_R(f \oplus \mathbf{1}_Q \mid P \oplus Q).$$

It is easy to check (Schanuel's Lemma !) that the definition above does not depend on Q . Now, one can use the same strategy to define the characteristic polynomial $\det_R(1 - f \cdot X \mid P) \in R[X]$ of variable X . Indeed, $P \otimes_R R[X]$ is a finitely generated, projective $R[X]$ -module. One defines

$$\det_R(1 - f \cdot X \mid P) := \det_{R[X]}(1 - f \otimes X \mid P \otimes_R R[X]).$$

Obviously, for any P , R and f as above and any R -algebra R' , we have base-change equalities

$$(5) \quad \begin{aligned} \det_R(f \mid P) &= \det_{R'}(f \otimes \mathbf{1}_{R'} \mid P \otimes_R R'), \\ \det_R(1 - f \cdot X \mid P) &= \det_{R'}(1 - (f \otimes \mathbf{1}_{R'}) \cdot X \mid P \otimes_R R'). \end{aligned}$$

Now, for any R as above and any finitely presented R -module M , the first Fitting invariant (ideal) $\text{Fit}_R(M)$ of M over R is defined as follows. First, one considers a finite presentation of M

$$R^n \xrightarrow{\phi} R^m \longrightarrow M \longrightarrow 0.$$

By definition, the Fitting ideal $\text{Fit}_R(M)$ is the ideal in R generated by the determinants of all the $m \times m$ minors of the matrix A_ϕ associated to ϕ with respect to two R -bases of R^n and R^m . It is well-known that the definition does not depend on the chosen presentation or bases, and

$$\text{Ann}_R(M)^m \subseteq \text{Fit}_R(M) \subseteq \text{Ann}_R(M).$$

For more on Fitting ideals, the reader can consult the first sections of [17] and the references therein.

Next, we state and prove a proposition which is of independent interest and will play an important role in our Fitting ideal calculations. In what follows, if R is a commutative topological ring and \mathcal{G} is a commutative profinite group, then the profinite group algebra

$$R[[\mathcal{G}]] := \varprojlim_{\mathcal{H}} R[\mathcal{G}/\mathcal{H}],$$

where \mathcal{G}/\mathcal{H} are all the finite quotients of \mathcal{G} by closed subgroups \mathcal{H} , is viewed as a topological R -algebra endowed with the usual projective limit topology.

Proposition 4.1. *Let R be a semi-local compact topological ring and \mathcal{G} a pro-cyclic group of topological generator g . Let M be a topological $R[[\mathcal{G}]]$ -module, which is projective and finitely generated as an R -module. Let $F(u) := \det_R(1 - \mu_g \cdot u \mid M)$, where μ_g is the $R[[\mathcal{G}]]$ -automorphism of M given by multiplication by g . Then, M is finitely presented as an $R[[\mathcal{G}]]$ -module and we have an equality of $R[[\mathcal{G}]]$ -ideals*

$$\text{Fit}_{R[[\mathcal{G}]]}(M) = (F(g^{-1})).$$

Proof. Since R is semi-local and Fitting ideals and characteristic polynomials are well behaved with respect to base-change, we may work on each local component separately. Consequently, we may assume that R is local and M is R -free of finite rank. Let $\{x_1, \dots, x_n\}$ be an R -basis for M and let $A_g \in \text{GL}_n(R)$ be the matrix associated to μ_g in this basis. We denote by ϕ_g the

$R[[\mathcal{G}]]$ -linear endomorphism of $R[[\mathcal{G}]]^n$ whose matrix in the standard $R[[\mathcal{G}]]$ -basis $\{e_1, \dots, e_n\}$ is $(1 - g^{-1} \cdot A_g) \in M_n(R[[\mathcal{G}]])$. We claim that we have an exact sequence of topological $R[[\mathcal{G}]]$ -modules

$$(6) \quad R[[\mathcal{G}]]^n \xrightarrow{\phi_g} R[[\mathcal{G}]]^n \xrightarrow{\pi} M \longrightarrow 0,$$

where π is the $R[[\mathcal{G}]]$ -linear morphism satisfying $\pi(e_i) = x_i$, for all $i = 1, \dots, n$. By definition, π is surjective and $\pi \circ \phi_g = 0$. So, the only thing that needs to be checked is the inclusion $\ker \pi \subseteq \text{Im } \phi_g$. In order to prove this, let $\iota : R^n \longrightarrow R[[\mathcal{G}]]^n$ be the canonical (R -linear) inclusion. Note that if $\{\bar{e}_1, \dots, \bar{e}_n\}$ is the standard R -basis of R^n , then $\iota(\bar{e}_i) = e_i$, for all i . Since M is R -free of basis $\{x_1, \dots, x_n\}$, the compositum $\pi \circ \iota$ is an isomorphism of R -modules. Consequently, we have $\text{Im } \iota \cap \ker \pi = 0$. Further, we claim that we have an equality

$$(7) \quad \text{Im } \iota + \text{Im } \phi_g = R[[\mathcal{G}]]^n.$$

In order to prove this last claim, for all $m \in \mathbb{N}$ we let $\phi_{g,m}$ be the $R[[\mathcal{G}]]$ -module endomorphism of $R[[\mathcal{G}]]^n$ whose matrix in the basis $(e_i)_i$ is $(g^m \cdot I_n - A_g^m) \in M_n(R[[\mathcal{G}]])$. It is easily seen that for all m we have $\phi_{g,m} = \phi_g \circ \alpha_{g,m}$, for an easily computable $\alpha_{g,m} \in \text{End}_{R[[\mathcal{G}]]}(R[[\mathcal{G}]]^n)$. This shows that $\text{Im } \phi_{g,m} \subseteq \text{Im } \phi_g$, for all m . Since, A_g has entries in R , this implies further that $g^m \cdot e_i \in \text{Im } \phi_g + \text{Im } \iota$, for all m and all i . Consequently, we have an inclusion

$$\bigoplus_{i=1}^n R[[\mathcal{G}]] \cdot e_i \subseteq \text{Im } \phi_g + \text{Im } \iota.$$

Now, we take the topological closure in $R[[\mathcal{G}]]^n$ of both sides of the above inclusion. Since R^n and $R[[\mathcal{G}]]^n$ are compact and ι and ϕ_g are continuous, the right-hand side is already closed. Moreover, in the profinite topology, $R[[\mathcal{G}]]$ is dense in $R[[\mathcal{G}]]$. Consequently, we have

$$R[[\mathcal{G}]]^n = \bigoplus_{i=1}^n R[[\mathcal{G}]] \cdot e_i \subseteq \text{Im } \phi_g + \text{Im } \iota,$$

which concludes the proof of our last claim (7). Now, the desired inclusion $\ker \pi \subseteq \text{Im } \phi_g$ is easily obtained by combining

$$\text{Im } \iota + \text{Im } \phi_g = R[[\mathcal{G}]]^n, \quad \text{Im } \phi_g \subseteq \ker \pi, \quad \text{and } \text{Im } \iota \cap \ker \pi = 0.$$

This concludes the proof of the fact that (6) is indeed an exact sequence of $R[[\mathcal{G}]]$ -modules. On one hand, this implies that M is indeed $R[[\mathcal{G}]]$ -finitely presented. On the other hand, from the definition of the Fitting ideal, we have an equality of $R[[\mathcal{G}]]$ -ideals

$$\text{Fit}_{R[[\mathcal{G}]]}(M) = (\det(1 - g^{-1} \cdot A_g)) = (F(g^{-1})).$$

This concludes the proof of our Proposition. □

Corollary 4.2. *Let R , \mathcal{G} and M be as in Proposition 4.1. The following hold.*

- (1) *Let $M_R^* := \text{Hom}_R(M, R)$. We view M_R^* as a topological $R[[\mathcal{G}]]$ -module with the co-variant \mathcal{G} -action, given by $\sigma \cdot f(m) := f(\sigma \cdot m)$, for all $f \in M_R^*$, $\sigma \in \mathcal{G}$ and $m \in M$. Then, we have*

$$\text{Fit}_{R[[\mathcal{G}]]}(M) = \text{Fit}_{R[[\mathcal{G}]]}(M_R^*).$$

- (2) *Further, assume $R = \mathbb{Z}_\ell[G]$, where G is a finite, abelian group. Let $M^* := \text{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell)$, viewed as an $R[[\mathcal{G}]]$ -module with the co-variant $G \times \mathcal{G}$ -action. Then*

$$\text{Fit}_{R[[\mathcal{G}]]}(M^*) = \text{Fit}_{R[[\mathcal{G}]]}(M).$$

Proof. (1) As in the proof of Proposition 4.1, we may assume that M is a free R -module of basis $\bar{x} := \{x_1, \dots, x_n\}$. Then M_R^* is R -free of basis $\bar{x}^* := \{x_1^*, \dots, x_n^*\}$, uniquely characterized by $x_i^*(x_j) = \delta_{i,j}$, for all $i, j = 1, \dots, n$. If one denotes by μ_g^* the multiplication by g map on M_R^* and by A_g^* its matrix with respect to the basis \bar{x}^* , it is easy to see that, under the co-variant \mathcal{G} -action, we have $A_g^* = A_g^t$, where A_g^t is the transposed of A_g . Consequently, we have

$$F^*(u) := \det_R(1 - \mu_g^* \cdot u \mid M_R^*) = \det_R(1 - \mu_g \cdot u \mid M) = F(u).$$

Now, part (1) of the corollary follows by applying Proposition 4.1 to M and M_R^* .

Part (2) follows immediately from (1) and the isomorphism of $R[[\mathcal{G}]]$ -modules $M^* \xrightarrow{\sim} M_R^*$ sending $\phi \in M^*$ to $\psi \in M_R^*$, with $\psi(x) = \sum_{\sigma \in G} \phi(x^{\sigma^{-1}}) \cdot \sigma$, for all $x \in M$. \square

4.2. The statement of the problem. Now, we are ready to state precisely the number theoretic task described in the introduction to this section. For that purpose, we make a slight change in notations, in order to be in tune with the more number theoretically minded reader. We let K/k denote an abelian G -Galois extension of characteristic p global fields. Assume that \mathbb{F}_q is the exact field of constants in k (but not necessarily in K .) Let $X \mapsto Y$ be the corresponding G -Galois cover of smooth projective curves defined over \mathbb{F}_q . Let S and Σ be two finite, nonempty, disjoint sets of closed points on Y , such that S contains the set S_{ram} of points which ramify in X . We let \mathbb{F} denote the algebraic closure of \mathbb{F}_q , $\overline{X} := X \times_{\mathbb{F}_q} \mathbb{F}$, $\overline{X} := X \times_{\mathbb{F}_q} \mathbb{F}$. Also, \overline{S} and $\overline{\Sigma}$ denote the sets of closed points on \overline{X} sitting above points in S and Σ , respectively. As in the last section, we let $H := \text{Gal}(K/k \cap k_{\infty})$, where $k_{\infty} := k\mathbb{F}$. For every closed point v on Y which does not belong to S_{ram} , we denote by G_v and σ_v the decomposition group and Frobenius automorphism associated to v inside G , respectively. Also, for a closed point v of Y , we let d_v denote its residual degree over \mathbb{F}_q (i.e. the degree over \mathbb{F}_q of the residue field associated to v) and we let $Nv := q^{d_v} = |\mathbb{F}_{q^{d_v}}|$ (i.e. the cardinality of the residue field associated to v .)

To the set of data $(K/k, \mathbb{F}_q, S, T)$, one can associate a (polynomial) equivariant L -function

$$(8) \quad \Theta_{S, \Sigma}(u) := \prod_{v \in \Sigma} (1 - \sigma_v^{-1} \cdot (qu)^{d_v}) \cdot \prod_{v \notin S} (1 - \sigma_v^{-1} \cdot u^{d_v})^{-1},$$

where the infinite product on the right is taken over all closed points in Y which are not in S . The product on the right-hand side is convergent in $\mathbb{Z}[G][[u]]$ and in fact it converges to an element $\Theta_{S, \Sigma}(u) \in \mathbb{Z}[G][u]$ (see [20], Chpt. V.) The link between $\Theta_{S, \Sigma}(u)$ and the classical Artin L -functions associated to the characters (irreducible representations) of the Galois group G is as follows. For every complex valued (irreducible) character χ of G , we let $L_{S, \Sigma}(\chi, s)$ denote the (S, Σ) -modified Artin L -function associated to χ . This is the (unique) holomorphic function of complex variable s , satisfying the equality

$$L_{S, \Sigma}(\chi, s) = \prod_{v \in \Sigma} (1 - \chi(\sigma_v) \cdot Nv^{1-s}) \cdot \prod_{v \notin S} (1 - \chi(\sigma_v) \cdot Nv^{-s})^{-1}, \quad \text{for all } s \in \mathbb{C}, \text{ with } \Re(s) > 1.$$

It is not difficult to show that we have an equality

$$\Theta_{S, \Sigma}(q^{-s}) = \sum_{\chi} L_{S, \Sigma}(\chi, s) \cdot e_{\chi^{-1}}, \quad \text{for all } s \in \mathbb{C},$$

where the sum is taken with respect to all the irreducible complex valued characters χ of G and $e_{\chi} := 1/|G| \sum_{\sigma \in G} \chi(\sigma) \sigma^{-1}$ denotes the idempotent corresponding to χ in $\mathbb{C}[G]$.

We denote by $M_{\overline{S}, \overline{\Sigma}}$ the Picard 1-motive associated to the set of data $(\overline{X}, \mathbb{F}, \overline{S}, \overline{\Sigma})$. For a prime number ℓ we consider the ℓ -adic Tate module (ℓ -adic realization) $T_\ell(M_{\overline{S}, \overline{\Sigma}})$ of $M_{\overline{S}, \overline{\Sigma}}$, endowed with the usual $\mathbb{Z}_\ell[[\overline{G}]]$ -module structure, where $\overline{G} := G \times \Gamma$ and $\Gamma := \text{Gal}(\mathbb{F}/\mathbb{F}_q)$ (see Remark 2.10.) We denote by γ the q -power arithmetic Frobenius, which is the distinguished topological generator of Γ characterized by $\gamma(\zeta) = \zeta^q$, for all $\zeta \in \mathbb{F}$. Theorem 3.10 restricted to this context assures us that $T_\ell(M_{\overline{S}, \overline{\Sigma}})$ is $\mathbb{Z}_\ell[G]$ -projective of finite rank. Proposition 4.1 combined with Remark 2.10 implies that $T_\ell(M_{\overline{S}, \overline{\Sigma}})$ is finitely presented as a (topological) $\mathbb{Z}_\ell[[\overline{G}]]$ -module, for all prime numbers ℓ . The main goal of this section is to prove part (2) of the following.

Theorem 4.3. *Under the above hypotheses, the following hold for all prime numbers ℓ .*

- (1) *The $\mathbb{Z}_\ell[G]$ -module $T_\ell(M_{\overline{S}, \overline{\Sigma}})$ is projective.*
- (2) *We have an equality of $\mathbb{Z}_\ell[[\overline{G}]]$ -ideals*

$$(\Theta_{S, \Sigma}(\gamma^{-1})) = \text{Fit}_{\mathbb{Z}_\ell[[\overline{G}]]}(T_\ell(M_{\overline{S}, \overline{\Sigma}})).$$

Of course, only (2) requires a proof. This task will be accomplished in two steps, the first (and the easier) step dealing with primes $\ell \neq p$, and the second dealing with the characteristic prime p .

4.3. The case $\ell \neq p$. We work under the assumptions and with the notations introduced in the previous subsection. The following theorem provides the link between the ℓ -adic realization $T_\ell(M_{\overline{S}, \overline{\Sigma}})$ of the Picard 1-motive $M_{\overline{S}, \overline{\Sigma}}$ and the G -equivariant L -function $\Theta_{S, \Sigma}(u)$.

Theorem 4.4. (Deligne) *For all primes $\ell \neq p$, the following equality holds in $\mathbb{Q}_\ell[G]$.*

$$\Theta_{S, \Sigma}(u) = \det_{\mathbb{Q}_\ell[G]}(1 - \gamma \cdot u \mid T_\ell(M_{\overline{S}, \overline{\Sigma}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

Proof. See Chpt. V of [20]. □

Now, we are ready to state and prove the main result of this subsection.

Theorem 4.5. *Under the above hypotheses, the following hold for all primes $\ell \neq p$.*

- (1) *The $\mathbb{Z}_\ell[G]$ -module $T_\ell(M_{\overline{S}, \overline{\Sigma}})$ is projective.*
- (2) *We have the following equality of ideals in $\mathbb{Z}_\ell[[\overline{G}]]$.*

$$(\Theta_{S, \Sigma}(\gamma^{-1})) = \text{Fit}_{\mathbb{Z}_\ell[[\overline{G}]]}(T_\ell(M_{\overline{S}, \overline{\Sigma}})).$$

Proof. Obviously, part (1) above is a particular case of Theorem 3.10. (Set $\kappa_0 := \mathbb{F}_q$ and $\kappa := \mathbb{F}$.) Now, part (1) above and the base-change equalities (5) permit us to base-change from $\mathbb{Q}_\ell[G]$ down to $\mathbb{Z}_\ell[G]$ in the statement of Theorem 4.4, which leads to an equality

$$\Theta_{S, \Sigma}(u) = \det_{\mathbb{Z}_\ell[G]}(1 - \gamma \cdot u \mid T_\ell(M_{\overline{S}, \overline{\Sigma}})).$$

Finally, part (2) of the statement is a direct consequence of the equality above and Proposition 4.1 applied to $M := T_\ell(M_{\overline{S}, \overline{\Sigma}})$, $R := \mathbb{Z}_\ell[G]$, $\mathcal{G} := \Gamma$, and $g := \gamma$. Note that R is indeed compact in the ℓ -adic topology and semi-local (of local direct summands $\mathbb{Z}_\ell[\chi][L]$, where χ runs through a complete set of representatives for $G(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ -conjugacy classes of $\overline{\mathbb{Q}_\ell}$ -valued irreducible characters of G , and L is the ℓ -Sylow subgroup of G .) □

4.4. The case $\ell = p$. Compared to the case $\ell \neq p$, the main difficulty in this case stems from the fact that the equality in Theorem 4.4 does not hold for $\ell = p$. Roughly speaking, this is due to the fact that $\text{rank}_{\mathbb{Z}_p} T_p(M_{\overline{S}, \overline{\Sigma}})$ is strictly smaller than $\text{rank}_{\mathbb{Z}_\ell} T_\ell(M_{\overline{S}, \overline{\Sigma}})$, for $\ell \neq p$, as follows immediately from Theorem 3.6 above. A more enlightening way of saying this is that the \mathbb{Z}_p -rank of p -adic étale cohomology in characteristic p is smaller than the \mathbb{Z}_ℓ -rank of ℓ -adic étale cohomology, for all $\ell \neq p$. (In fact, the modules $T_\ell(M_{\overline{S}, \overline{\Sigma}})$ can be identified with the functorial duals of the first ℓ -adic étale cohomology groups of $M_{\overline{S}, \overline{\Sigma}}$.) This is why, in order to obtain a p -adic equality similar to the one in Theorem 4.4, one has to replace p -adic étale cohomology with something larger (of the right rank), namely crystalline cohomology. We describe this next.

If $W := W(\mathbb{F})$ denotes the ring of Witt vectors associated to \mathbb{F} , then we let $H_{\text{cris}}^1 := H_{\text{cris}}^1(\overline{X}/W)$ denote the first crystalline cohomology group associated to \overline{X} . We denote by $H_p^1 := H_{\text{ét}}^1(\overline{X}, \mathbb{Z}_p)$ the first p -adic étale cohomology group associated to \overline{X} . For the properties of crystalline and étale cohomology relevant in this context, the reader can consult the Appendix of [17] and the references therein. In particular, we remind the reader that these are free, finite rank W -modules and \mathbb{Z}_p -modules, respectively, on which the Galois group G acts naturally. The q -power geometric Frobenius endomorphism F associated to \overline{X} induces a $W[G]$ -module endomorphism and a $\mathbb{Z}_p[G]$ -module endomorphisms (both denoted by F^*) of H_{cris}^1 and H_p^1 , respectively. An important theorem of Bloch and Illusie (see II.5.4 in [12], Lemma 3.3 in [4], or Appendix of [17]) identifies H_p^1 with a $\mathbb{Z}_p[G]$ -submodule of H_{cris}^1 , such that

$$(9) \quad H_p^1 \otimes_{\mathbb{Z}_p} \mathbb{C}_p = (H_{\text{cris}}^1 \otimes_W \mathbb{C}_p)_0,$$

where $(H_{\text{cris}}^1 \otimes_W \mathbb{C}_p)_0$ is the $\mathbb{C}_p[G]$ -submodule of $H_{\text{cris}}^1 \otimes_W \mathbb{C}_p$ on which F^* acts with eigenvalues which are p -adic units (i.e. of p -adic valuation equal to 0.) We have a direct sum decomposition

$$(10) \quad H_{\text{cris}}^1 \otimes_W \mathbb{C}_p = (H_{\text{cris}}^1 \otimes_W \mathbb{C}_p)_0 \oplus (H_{\text{cris}}^1 \otimes_W \mathbb{C}_p)_{>0}$$

in the category of $\mathbb{C}_p[G]$ -modules, where $(H_{\text{cris}}^1 \otimes_W \mathbb{C}_p)_{>0}$ is the $\mathbb{C}_p[G]$ -submodule of $H_{\text{cris}}^1 \otimes_W \mathbb{C}_p$ on which F^* acts with eigenvalues of strictly positive p -adic valuation.

When dealing with the G -equivariant (polynomial) L -function $\Theta_{S, \Sigma}(u)$, it is more convenient to work with the functorial duals $H_{1, \text{cris}}$ and $H_{1, p}$ of H_{cris}^1 and H_p^1 , respectively (i.e. the homology groups.) The (contravariant) action of the geometric Frobenius F^* on the cohomology groups turns into the (covariant) action of the arithmetic Frobenius $F_* := \gamma$ on the homology groups. Of course, we have an equality and direct sum decomposition similar to (9) and (10), respectively, with homology replacing co-homology and the subscripts “0” and “ > 0 ” referring to the p -adic valuations of the eigenvalues of γ acting on homology. We remind the reader that there is a canonical isomorphism ($\mathbb{Z}_p[G]$ -linear and preserving the γ -action on both sides)

$$(11) \quad H_{1, p} \xrightarrow{\sim} T_p(J_{\overline{X}}),$$

where $T_p(J_{\overline{X}})$ denotes the p -adic Tate module of the Jacobian $J_{\overline{X}}$ associated to \overline{X} (see [17].) The p -adic homological interpretation of $\Theta_{S, \Sigma}(u)$ is given by the following theorem, essentially due to Berthelot (see the Appendix of [17] and §2.5 of [11].)

Theorem 4.6. *The following equality holds true in $\mathbb{C}_p[G][u]$ (det always taken over $\mathbb{C}_p[G]$).*

$$\Theta_{S, \Sigma}(u) = \frac{\prod_{v \in \Sigma} (1 - \sigma_v^{-1} \cdot (qu)^{d_v})}{\det(1 - \gamma \cdot qu \mid \mathbb{C}_p[G/H])} \cdot \det(1 - \gamma \cdot u \mid H_{1, \text{cris}} \otimes_W \mathbb{C}_p) \cdot \det(1 - \gamma u \mid \text{Div}^0(\overline{S}) \otimes \mathbb{C}_p),$$

where γ acts on $\mathbb{C}_p[G/H]$ by multiplication with the inverse of the canonical generator $\overline{\gamma}$ of G/H .

Remark 4.7. Note that if we let $\pi : G \rightarrow G/H$ denote the canonical projection, then $\pi(\sigma_v) = \bar{\gamma}^{d_v}$. This implies right away that we have a divisibility in $\mathbb{C}_p[G][u]$

$$\det_{\mathbb{C}_p[G]}(1 - \gamma \cdot qu \mid \mathbb{C}_p[G/H]) \mid (1 - \sigma_v^{-1} \cdot (qu)^{d_v}),$$

for all $v \in \Sigma$. Consequently, we have

$$\frac{\prod_{v \in \Sigma} (1 - \sigma_v^{-1} \cdot (qu)^{d_v})}{\det_{\mathbb{C}_p[G]}(1 - \gamma \cdot qu \mid \mathbb{C}_p[G/H])} \in \mathbb{C}_p[G][u].$$

Next, we express the right-hand side of the equality in the Theorem above as a product of two polynomials $P, Q \in \mathbb{C}_p[G][u]$, which we define below.

$$\begin{aligned} P(u) &:= \det_{\mathbb{C}_p[G]}(1 - \gamma \cdot u \mid (H_{1,\text{cris}} \otimes_W \mathbb{C}_p)_0) \cdot \det_{\mathbb{C}_p[G]}(1 - \gamma u \mid \text{Div}^0(\bar{S}) \otimes \mathbb{C}_p), \\ Q(u) &:= \frac{\prod_{v \in \Sigma} (1 - \sigma_v^{-1} \cdot (qu)^{d_v})}{\det_{\mathbb{C}_p[G]}(1 - \gamma \cdot qu \mid \mathbb{C}_p[G/H])} \cdot \det_{\mathbb{C}_p[G]}(1 - \gamma \cdot u \mid (H_{1,\text{cris}} \otimes_W \mathbb{C}_p)_{>0}). \end{aligned}$$

Proposition 4.8. With notations as above, we have the following.

- (1) $P(u) = \det_{\mathbb{Z}_p[G]}(1 - \gamma u \mid T_p(M_{\bar{S}, \Sigma}))$.
- (2) $P(u) \in \mathbb{Z}_p[G][u]$.
- (3) $Q(u) \in \mathbb{Z}_p[G][u]$.

Proof. If we combine (11) above with the definition of P , we obtain the following equalities.

$$\begin{aligned} P(u) &= \det_{\mathbb{C}_p[G]}(1 - \gamma \cdot u \mid H_{1,p} \otimes_{\mathbb{Z}_p} \mathbb{C}_p) \cdot \det_{\mathbb{C}_p[G]}(1 - \gamma u \mid \text{Div}^0(\bar{S}) \otimes \mathbb{C}_p) \\ &= \det_{\mathbb{C}_p[G]}(1 - \gamma \cdot u \mid T_p(J_{\bar{X}}) \otimes_{\mathbb{Z}_p} \mathbb{C}_p) \cdot \det_{\mathbb{C}_p[G]}(1 - \gamma u \mid \text{Div}^0(\bar{S}) \otimes \mathbb{C}_p) \\ &= \det_{\mathbb{C}_p[G]}(1 - \gamma \cdot u \mid T_p(M_{\bar{S}, \emptyset}) \otimes_{\mathbb{Z}_p} \mathbb{C}_p). \end{aligned}$$

However, Remark 2.7 shows that we have an equality of $\mathbb{Z}_p[G]$ -modules

$$T_p(M_{\bar{S}, \Sigma}) = T_p(M_{\bar{S}, \emptyset}),$$

and Theorem 3.10 shows that these modules are $\mathbb{Z}_p[G]$ -projective of finite rank. Therefore, we have

$$\det_{\mathbb{C}_p[G]}(1 - \gamma \cdot u \mid T_p(M_{\bar{S}, \emptyset}) \otimes_{\mathbb{Z}_p} \mathbb{C}_p) = \det_{\mathbb{Z}_p[G]}(1 - \gamma u \mid T_p(M_{\bar{S}, \Sigma})),$$

which proves part (1). (Note that we have applied equalities (5) to base-change from $\mathbb{C}_p[G]$ down to $\mathbb{Z}_p[G]$.) Of course, part (2) is a consequence of part (1). In order to prove part (3), recall the following elementary result.

Lemma 4.9. Let R be a subring of the commutative ring R' with 1. Let $P, Q \in S[u]$ be two polynomials, such that $P(0) = 1$, $P \in R[u]$ and $P \cdot Q \in R[u]$. Then $Q \in R[u]$.

Proof. See Lemma A.5 in the Appendix of [17]. □

Now, apply the Lemma above to our polynomials P and Q and rings $R := \mathbb{Z}_p[G]$ and $R' := \mathbb{C}_p[G]$ and keep in mind that $P \in R[u]$ (which is (2) above), $P(0) = 1$ (from the definition of P as characteristic polynomial) and that $P \cdot Q = \Theta_{S, \Sigma}(u) \in \mathbb{Z}[G][u] \subseteq \mathbb{Z}_p[G][u]$. This concludes the proof of (3). □

Our next goal is to prove the following.

Proposition 4.10. We have $Q(\gamma^{-1}) \in \mathbb{Z}_p[[\bar{G}]]^\times$.

Proof. We need the following elementary Lemma.

Lemma 4.11. *Let \mathcal{G} be a profinite abelian group and let x be an element in the profinite group ring $\mathbb{Z}_p[[\mathcal{G}]]$. The following are equivalent.*

- (1) $x \in \mathbb{Z}_p[[\mathcal{G}]]^\times$.
- (2) *For every continuous p -adic character ψ of \mathcal{G} , we have $\psi(x) \in O_\psi^\times$, where O_ψ denotes the finite extension of \mathbb{Z}_p generated by the values of ψ .*

Proof. We have $\mathbb{Z}_p[[\mathcal{G}]] = \varprojlim \mathbb{Z}_p[[\mathcal{G}/\mathcal{H}]]$ and therefore $\mathbb{Z}_p[[\mathcal{G}]]^\times = \varprojlim \mathbb{Z}_p[[\mathcal{G}/\mathcal{H}]]^\times$, where the projective limits are taken with respect to all the subgroups \mathcal{H} of finite index in \mathcal{G} . Since every continuous character of \mathcal{G} factors through a character of \mathcal{G}/\mathcal{H} , for some \mathcal{H} as above, it suffices to show the equivalence of (1) and (2) in the situation where \mathcal{G} is finite. However, in the case where \mathcal{G} is finite, we have an equality $\mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[\mathcal{G}]$ and an injective ring morphism

$$\mathbb{Z}_p[\mathcal{G}] \longrightarrow \bigoplus_\psi O_\psi, \quad y \longrightarrow \bigoplus_\psi \psi(y),$$

which induces an integral extension of rings, where the sum is taken with respect to all the irreducible \mathbb{C}_p -valued characters ψ of \mathcal{G} . Since it is well known (and quite elementary) that in an integral ring extension $\mathcal{A} \subseteq \mathcal{B}$ we have $x \in \mathcal{A}^\times$ if and only if $x \in \mathcal{B}^\times$, for all $x \in \mathcal{A}$, the equivalence of (1) and (2) is established. \square

Now, we are ready to prove Proposition 4.10. First, let us note that any continuous p -adic character ψ of $\overline{G} = G \times \Gamma$ is a product $\psi = (\chi, \rho)$, where χ is a p -adic character of G and ρ is a continuous p -adic character of Γ . Therefore, we have

$$\psi(Q(\gamma^{-1})) = Q^\chi(\rho(\gamma^{-1})),$$

where $Q^\chi(u)$ is the polynomial in $O_\chi[u]$ obtained by evaluating χ at the coefficients of $Q(u) \in \mathbb{Z}_p[G][u]$. In order to simplify notation, let us write $Q(u) = Q_\Sigma(u) \cdot Q_{\text{cris}}(u)$, where

$$Q_\Sigma(u) := \frac{\prod_{v \in \Sigma} (1 - \sigma_v^{-1} \cdot (qu)^{d_v})}{\det_{\mathbb{C}_p[G]}(1 - \gamma \cdot qu \mid \mathbb{C}_p[G/H])}, \quad Q_{\text{cris}}(u) := \det_{\mathbb{C}_p[G]}(1 - \gamma \cdot u \mid (H_{1,\text{cris}} \otimes_W \mathbb{C}_p)_{>0}).$$

Note that in general we have $Q_\Sigma(u), Q_{\text{cris}}(u) \in \mathbb{C}_p[G][u] \setminus \mathbb{Z}_p[G][u]$, but $Q(u) \in \mathbb{Z}_p[G][u]$ (see Proposition 4.8.) Now, let $\psi = (\chi, \rho)$ be a character as above. Remark 4.7 implies that we have

$$Q_\Sigma^\chi(u) = \begin{cases} \prod_{v \in \Sigma} (1 - \chi(\sigma_v^{-1}) \cdot (qu)^{d_v}), & \text{if } \chi|_H \neq 1_H; \\ \left(\sum_{i=0}^{d_{v_0}-1} \chi^i(\overline{\gamma}^{-1}) q^i \cdot u^i \right) \cdot \prod_{v \in \Sigma \setminus \{v_0\}} (1 - \chi(\sigma_v^{-1}) \cdot (qu)^{d_v}), & \text{if } \chi|_H = 1_H. \end{cases}$$

$$Q_{\text{cris}}^\chi(u) = \prod_{i=1}^{d_\chi} (1 - \alpha_{i,\chi} \cdot u),$$

where $v_0 \in \Sigma$ is arbitrary, and the $\alpha_{i,\chi}$'s are the eigenvalues of γ acting on the d_χ -dimensional \mathbb{C}_p -vector space $(H_{1,\text{cris}} \otimes_W \mathbb{C}_p)_{>0}^\chi$, which is the χ -eigenspace of $(H_{1,\text{cris}} \otimes_W \mathbb{C}_p)_{>0}$ with respect to the action of G . Now, let O denote a large enough finite integral extension of W inside \mathbb{C}_p , containing the values of χ and ρ and the eigenvalues $\alpha_{i,\chi}$ for all $i = 1, \dots, d_\chi$. Note that since γ acts W -linearly on the free, finite rank W -module $H_{1,\text{cris}}$, the eigenvalues $\alpha_{i,\chi}$ are integral over W , for all χ and $i = 1, \dots, d_\chi$. Further, if we let \mathfrak{m}_O denote the maximal ideal of O , by the definition

of $(H_{1,\text{cris}} \otimes_W \mathbb{C}_p)_{>0}$, we have $\alpha_{i,\chi} \in \mathfrak{m}_O$, for all χ and i as above. This observation combined with the obvious fact that $q \in \mathfrak{m}_O$ implies that

$$Q_\Sigma^\chi(u), Q_{\text{cris}}^\chi(u), Q^\chi(u) \in 1 + u \cdot \mathfrak{m}_O[u].$$

Consequently, we have

$$\psi(Q(\gamma^{-1})) = Q^\chi(\rho(\gamma^{-1})) \in 1 + \mathfrak{m}_O \subseteq O^\times.$$

However, since $Q(\gamma^{-1}) \in \mathbb{Z}_p[[\overline{G}]]$, we know that $\psi(Q(\gamma^{-1})) \in O_\psi$. Consequently,

$$\psi(Q(\gamma^{-1})) \in O_\psi \cap O^\times = O_\psi^\times.$$

According to the Lemma above, this concludes the proof of Proposition 4.10. \square

Now, we are ready to state and prove the main result of this subsection.

Theorem 4.12. *Under the above hypotheses, the following hold.*

- (1) *The $\mathbb{Z}_p[G]$ -module $T_p(M_{\overline{S}, \overline{\Sigma}})$ is projective.*
- (2) *We have the following equality of ideals in $\mathbb{Z}_p[[\overline{G}]]$.*

$$(\Theta_{S,\Sigma}(\gamma^{-1})) = \text{Fit}_{\mathbb{Z}_p[[\overline{G}]]}(T_p(M_{\overline{S}, \overline{\Sigma}})).$$

Proof. As stated in the proof of Proposition 4.8, part (1) above is a particular case of Theorem 3.10 (the case where $\kappa_0 = \mathbb{F}_q$.) Now, Propositions 4.8(1) and 4.10 imply that we have the following equalities of ideals in $\mathbb{Z}_p[[\overline{G}]]$.

$$(12) \quad (\Theta_{S,\Sigma}(\gamma^{-1})) = (P(\gamma^{-1})) = (\det_{\mathbb{Z}_p[G]}(1 - \gamma \cdot u \mid T_p(M_{\overline{S}, \overline{\Sigma}})) \mid_{u=\gamma^{-1}}).$$

Now, part (2) of our Theorem 4.12 is a direct consequence of equalities (12) and Proposition 4.1 applied to $M := T_p(M_{\overline{S}, \overline{\Sigma}})$, $R := \mathbb{Z}_p[G]$, $\mathcal{G} := \Gamma$, and $g := \gamma$. Note that R is indeed compact in the p -adic topology and semi-local (see the argument which ends the proof of Theorem 4.5.) \square

Now, we combine Theorems 4.5 and 4.12 to obtain Theorem 4.3, which completes the task set at the beginning of this section.

Next, we derive a corollary, which will be useful in the next section. For that purpose, let $K_\infty := K\mathbb{F}$ (field compositum viewed inside some separable closure of K .) This is the field of rational functions of the smooth, projective, irreducible curve $X_\infty := X \times_{\mathbb{F}_r} \mathbb{F}$, where $\mathbb{F}_r := \mathbb{F} \cap K$ is the exact field of constants of K . Let $G_\infty := \text{Gal}(K_\infty/k)$. Since Γ is a free abelian pro-finite group, we have a (non-canonical) isomorphism $G_\infty \simeq H \times \Gamma$, where $H := \text{Gal}(K_\infty/k_\infty) \simeq \text{Gal}(K/K \cap k_\infty)$. On the other hand, Galois theory and the natural isomorphism $\Gamma := \text{Gal}(\mathbb{F}/\mathbb{F}_q) \simeq \text{Gal}(k_\infty/k)$ permit us to identify G_∞ with the subgroup of $\overline{G} := G \times \Gamma$, consisting of all (g, σ) , with $g \in G$ and $\sigma \in \Gamma$, such that $\pi_G(g) = \pi_\Gamma(\sigma)$, where $\pi_G : \text{Gal}(K/k) = G \twoheadrightarrow G/H = \text{Gal}(k_\infty \cap K/k)$ and $\pi_\Gamma : \Gamma = \text{Gal}(k_\infty/k) \twoheadrightarrow \text{Gal}(k_\infty \cap K/k) = G/H$ are the usual projections induced by Galois restriction. There is an exact sequence in the category of groups

$$1 \longrightarrow G_\infty \longrightarrow \overline{G} \longrightarrow G/H \longrightarrow 1,$$

where the injection sends $\tau \in G_\infty$ to $(\tau \mid_K, \tau \mid_{k_\infty})$ and the surjection sends (g, σ) in $G \times \Gamma$ to $\pi_G(g)\pi_\Gamma(\sigma)^{-1}$ in G/H . This leads to a canonical identification of $\mathbb{Z}_\ell[[G_\infty]]$ with a subring of $\mathbb{Z}_\ell[[\overline{G}]]$. Since for every prime v in k , which is unramified in K/k , we have $\pi_G(\sigma_v) = \pi_\Gamma(\gamma^{d_v})$, the product formula (8) shows that, under the above identification, we have

$$\Theta_{S,\Sigma}(\gamma^{-1}) \in \mathbb{Z}_\ell[[G_\infty]] \subseteq \mathbb{Z}_\ell[[\overline{G}]].$$

Now, we let $M_{S_\infty, \Sigma_\infty}$ be the Picard 1-motive associated to $(X_\infty, \mathbb{F}, S_\infty, \Sigma_\infty)$, where S_∞ and Σ_∞ are the sets of closed points on X_∞ sitting above points in S and Σ , respectively. Its ℓ -adic realizations $T_\ell(M_{S_\infty, \Sigma_\infty})$ are endowed with natural $\mathbb{Z}_\ell[[G_\infty]]$ -module structures, for all primes ℓ .

Corollary 4.13. *For every prime number ℓ , the following hold.*

- (1) $T_\ell(M_{S_\infty, \Sigma_\infty})$ is a projective $\mathbb{Z}_\ell[H]$ -module.
- (2) We have an equality of $\mathbb{Z}_\ell[[G_\infty]]$ -ideals $\text{Fit}_{\mathbb{Z}_\ell[[G_\infty]]}(T_\ell(M_{S_\infty, \Sigma_\infty})) = (\Theta_{S, \Sigma}(\gamma^{-1}))$.

Proof. The proof of Theorem 3.10 gives an isomorphism of $\mathbb{Z}_\ell[G]$ -modules

$$T_\ell(M_{\overline{S}, \overline{\Sigma}}) \simeq T_\ell(M_{S_\infty, \Sigma_\infty}) \otimes_{\mathbb{Z}_\ell[H]} \mathbb{Z}_\ell[G],$$

for all primes ℓ . This isomorphism combined with Theorem 4.3(1) implies part (1) of the Corollary. In order to prove part (2), let $R_\infty := \mathbb{Z}_\ell[[G_\infty]]$ and $\overline{R} := R[[\overline{G}]]$. Since $\overline{G}/G_\infty \simeq G/H$ (see above), \overline{R} is a free R_∞ -module of rank $|G/H|$. Consequently, \overline{R} is a faithfully flat R_∞ -algebra. The isomorphism above can be re-written as an isomorphism of \overline{R} -modules

$$T_\ell(M_{\overline{S}, \overline{\Sigma}}) \simeq T_\ell(M_{S_\infty, \Sigma_\infty}) \otimes_{R_\infty} \overline{R}.$$

Consequently, since Fitting ideals commute with extension of scalars, the isomorphism above combined with Theorem 4.3(2) gives an equality of \overline{R} -ideals

$$\text{Fit}_{R_\infty}(T_\ell(M_{S_\infty, \Sigma_\infty}))\overline{R} = \text{Fit}_{\overline{R}}(T_\ell(M_{\overline{S}, \overline{\Sigma}})) = \Theta_{S, \Sigma}(\gamma^{-1}) \cdot \overline{R}$$

Now, recall that since \overline{R} is a faithfully flat R_∞ -algebra, we have $I\overline{R} \cap R_\infty = I$, for all ideals I in R_∞ (see [14], Theorem 7.5(ii), p.49.) If we apply this property to the R_∞ -ideals $\text{Fit}_{R_\infty}(T_\ell(M_{S_\infty, \Sigma_\infty}))$ and $(\Theta_{S, \Sigma}(\gamma^{-1}))$ and take into account the equality of \overline{R} -ideals above, we obtain

$$\text{Fit}_{R_\infty}(T_\ell(M_{S_\infty, \Sigma_\infty})) = (\Theta_{S, \Sigma}(\gamma^{-1})) = \text{Fit}_{\overline{R}}(T_\ell(M_{\overline{S}, \overline{\Sigma}})) \cap R_\infty,$$

which concludes the proof of part (2). \square

5. Refinements of the Brumer-Stark and Coates-Sinnott conjectures for global fields of characteristic p .

In this section, we prove that Theorem 4.3 (or, more precisely, Corollary 4.13) above implies refinements of the Brumer-Stark and Coates-Sinnott conjectures, linking special values of equivariant Artin L -functions to certain Galois module structure invariants of ideal-class groups and ℓ -adic étale cohomology groups, in the context of abelian extensions of characteristic p global fields. We are working with the notations and under the hypotheses of the previous section.

5.1. Generalized Jacobians, class-field theory and ℓ -adic étale cohomology. If w is a closed point on X , we denote by $\mathbb{F}_r(w)$ its residue field and let $\deg(w) := [\mathbb{F}_r(w) : \mathbb{F}_r]$ denote its degree over \mathbb{F}_r . As customary, the degrees of divisors on X are computed over the strict field of constants \mathbb{F}_r of K . For every $n \in \mathbb{Z}_{\geq 1}$, we let K_n denote the field compositum $K_n := K \cdot F_{r^n}$, which is a characteristic p global field of exact field of constants \mathbb{F}_{r^n} . We let $\Gamma_K = \text{Gal}(K_\infty/K)$. Obviously, we have an isomorphism of topological groups $\Gamma_K \simeq \widehat{\mathbb{Z}}$. We denote by γ_K the canonical topological generator of Γ_K , given by the r -power arithmetic Frobenius. We let $X_n := X \times_{\mathbb{F}_r} \mathbb{F}_{r^n}$ denote the smooth, projective models of K_n over \mathbb{F}_{r^n} , for all n .

Definition 5.1. Let T be a finite (possibly empty) set of closed points in X . For $* \in \{\infty\} \cup \mathbb{N}$, we let T_* be the set of closed points on X_* sitting above points in T . We define the T -modified Picard groups associated to K_* (or to X_* , for that matter)

$$\text{Pic}_T(K_*) := \frac{\text{Div}(X_* \setminus T_*)}{\{\text{div}(f) \mid f \in K_{*,T}^\times\}}, \quad \text{Pic}_T^0(K_*) := \frac{\text{Div}^0(X_* \setminus T_*)}{\{\text{div}(f) \mid f \in K_{*,T}^\times\}},$$

where $K_{*,T}^\times := \{f \in K_*^\times \mid f \equiv 1 \pmod{w}, \text{ for all } w \in T_*\}$, as in Definition 2.1. (Obviously, $K_1 = K$ and $T_1 = T$.)

For $T = \emptyset$, one obtains the classical Picard groups and \emptyset will be dropped from the notation. We have an obvious commutative diagram with exact rows and columns

(13)

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{\bigoplus_{w \in T} \mathbb{F}_r(w)^\times}{\mathbb{F}_r^\times} & \longrightarrow & \text{Pic}_T^0(K) & \longrightarrow & \text{Pic}^0(K) \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \frac{\bigoplus_{w \in T} \mathbb{F}_r(w)^\times}{\mathbb{F}_r^\times} & \longrightarrow & \text{Pic}_T(K) & \longrightarrow & \text{Pic}(K) \longrightarrow 0 \\ & & \downarrow \text{deg} & & \downarrow \text{deg} & & \downarrow \\ & & \mathbb{Z} & = & \mathbb{Z} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

at the $K = K_1$ level and exact analogues at the K_* -levels, for all $*$ as above. The surjectivity of the degree maps (“deg”) at the finite levels is a classical theorem of F. K. Schmidt. If $T \neq \emptyset$, we view \mathbb{F}_r^\times as sitting inside $\bigoplus_{w \in T} \mathbb{F}_r(w)^\times$ diagonally. The injective maps in the horizontal short exact sequences send the class modulo \mathbb{F}_r^\times of an element $(x_w)_w \in \bigoplus_{w \in T} \mathbb{F}_r(w)^\times$ to the class of $\text{div}(f)$ of a function $f \in K^\times$ satisfying $f \equiv x_w \pmod{w}$, for all $w \in T$. The existence of f is implied by the weak approximation theorem.

Remark 5.2. Note that, for any T as above, we have $\text{Pic}_T^0(K_\infty) = J_{T_\infty}(\mathbb{F})$, where J_{T_∞} is the semi-abelian variety (generalized Jacobian) associated to the set of data (X_∞, T_∞) over \mathbb{F} (see §2).

Remark 5.3. For all T as above and all natural numbers m, n with $n \mid m$, the canonical maps

$$\text{Pic}_T^0(K_n) \longrightarrow \text{Pic}_T^0(K_m) \text{ and } \text{Pic}_T^0(K_n) \longrightarrow \text{Pic}_T^0(K_\infty)$$

are injective. Indeed, this follows immediately from the equalities $H^1(\Gamma_{m,n}, \mathbb{F}_{r^m}^\times) = H^1(\Gamma_n, \mathbb{F}^\times) = 1$, where $\Gamma_n := \text{Gal}(K_\infty/K_n)$ and $\Gamma_{m,n} := \text{Gal}(K_m/K_n)$. If we identify $\text{Pic}_T^0(K_n)$ with a subgroup of $\text{Pic}_T^0(K_\infty)$ under the above injective maps, then we have equalities

$$(14) \quad J_{T_\infty}(\mathbb{F})^{\Gamma_n} = \text{Pic}_T^0(K_\infty)^{\Gamma_n} = \text{Pic}_T^0(K_n), \quad J_{T_\infty}(\mathbb{F}) = \bigcup_n \text{Pic}_T^0(K_n).$$

Indeed, this is an immediate consequence of the fact that K_∞/K_n is everywhere unramified and $H^1(\Gamma_n, K_{\infty,T}^\times) = 1$, for all n . (See Step 2 in the proof of Theorem 3.1.)

Remark 5.4. *Global class-field theory establishes a canonical injective (Artin reciprocity) morphism*

$$\rho_{n,T} : \text{Pic}_T(K_n) \longrightarrow \mathfrak{X}_{n,T},$$

where $\mathfrak{X}_{n,T}$ is the Galois group of the maximal abelian extension $M_{n,T}$ of K_n which is unramified outside of T_n and at most tamely ramified at T_n . Moreover, the image of the above morphism is dense in $\mathfrak{X}_{n,T}$ and it consists of all $\sigma \in \mathfrak{X}_{n,T}$ which via the Galois restriction map $\mathfrak{X}_{n,T} \rightarrow \text{Gal}(K_\infty/K_n)$ land in the subgroup $(\gamma_K^n)^\mathbb{Z}$ of $\text{Gal}(K_\infty/K_n) = (\gamma_K^n)^\widehat{\mathbb{Z}}$. Consequently, the morphism above induces an isomorphism at the level of profinite completions

$$\widehat{\rho_{n,T}} : \widehat{\text{Pic}_T(K_n)} \xrightarrow{\sim} \mathfrak{X}_{n,T}.$$

Note that there is a (non-canonical) group isomorphism $\widehat{\text{Pic}_T(K_n)} \xrightarrow{\sim} \widehat{\text{Pic}_T^0(K_n)} \times \widehat{\mathbb{Z}}$, coming from a (non-canonical) splitting of the left-most vertical short exact sequence in the diagram (13) above.

Remark 5.5. Let $\mathfrak{X}_{\infty,T}$ denote the Galois group of the maximal abelian extension $M_{\infty,T}$ of K_∞ which is unramified away from T_∞ and at most tamely ramified at T_∞ . Since K_∞/K is unramified everywhere, it is easy to show that $M_{\infty,T} = \cup_n M_{n,T}$ (union viewed inside a fixed separable closure of K_∞ .) Consequently, we have an isomorphism of topological groups

$$\varprojlim_n \mathfrak{X}_{n,T} \xrightarrow{\sim} \mathfrak{X}_{\infty,T},$$

where the projective limit is taken with respect to the Galois restriction maps $\text{res}_{m,n} : \mathfrak{X}_{m,T} \rightarrow \mathfrak{X}_{n,T}$, for all n, m , such that $n \mid m$. Now, elementary properties of the Artin reciprocity map leads to a canonical isomorphism of topological groups

$$\varprojlim_n \widehat{\text{Pic}_T(K_n)} \xrightarrow{\sim} \mathfrak{X}_{\infty,T},$$

where the projective limit is taken with respect to the norm maps $N_{m,n} : \text{Pic}_T(K_m) \rightarrow \text{Pic}_T(K_n)$, for all m, n with $n \mid m$. However, as the reader can easily check, there are commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}_T^0(K_m) & \longrightarrow & \widehat{\text{Pic}_T(K_m)} & \xrightarrow{\widehat{\text{deg}}} & \widehat{\mathbb{Z}} & \longrightarrow 0 \\ & & \downarrow N_{m,n}^0 & & \downarrow \widehat{N_{m,n}} & & \downarrow \times \frac{m}{n} & \\ 0 & \longrightarrow & \text{Pic}_T^0(K_n) & \longrightarrow & \widehat{\text{Pic}_T(K_n)} & \xrightarrow{\widehat{\text{deg}}} & \widehat{\mathbb{Z}} & \longrightarrow 0 \end{array}$$

for all m and n as above, where $N_{m,n}^0$ denotes the restriction of $N_{m,n}$ to $\text{Pic}_T^0(K_m)$. Since $\varprojlim_n \widehat{\mathbb{Z}} = 0$,

we obtain canonical isomorphisms of topological groups

$$(15) \quad \varprojlim_n \text{Pic}_T^0(K_n) \xrightarrow{\sim} \varprojlim_n \widehat{\text{Pic}_T(K_n)} \xrightarrow{\sim} \mathfrak{X}_{\infty,T}.$$

Lemma 5.6. *Let M be a torsion, divisible \mathbb{Z}_ℓ -module of finite co-rank, endowed with a Γ_K -action which is continuous with respect to the discrete and profinite topology on M and Γ_K , respectively. Assume that M^{Γ_n} is finite, for all n . Then, the following hold.*

(1) *There exist canonical isomorphisms $\phi_n : M^{\Gamma_n} \xrightarrow{\sim} T_\ell(M)_{\Gamma_n}$ and commutative diagrams*

$$\begin{array}{ccc} M^{\Gamma_m} & \xrightarrow[\sim]{\phi_m} & T_\ell(M)_{\Gamma_m} \\ \downarrow N_{m,n} & & \downarrow \\ M^{\Gamma_n} & \xrightarrow[\sim]{\phi_n} & T_\ell(M)_{\Gamma_n}, \end{array}$$

for all n and m with $n \mid m$, where $N_{m,n}$ is the usual norm (multiplication by $\frac{\gamma_K^m - 1}{\gamma_K^n - 1}$) map and the vertical maps on the right are the natural projections.

(2) *There are natural isomorphisms of topological compact $\mathbb{Z}_\ell[[\Gamma_K]]$ -modules*

$$\varprojlim_n M^{\Gamma_n} \xrightarrow{\sim} \varprojlim_n T_\ell(M)_{\Gamma_n} \xrightarrow{\sim} T_\ell(M).$$

The first isomorphism above is the projective limit of the ϕ_n 's and the second is the inverse of the projective limit of the canonical surjections $\pi_n : T_\ell(M) \rightarrow T_\ell(M)_{\Gamma_n}$.

Proof. The proof is routine, except for two subtle points: the definition of the maps ϕ_n and the bijectivity of $\varprojlim_n \pi_n$. We will explain these two points, leaving the details to the reader.

• **Constructing ϕ_n .** Let us fix $n \in \mathbb{N}$ and $x \in M^{\Gamma_n}$. Since M is divisible, there exists

$$(x_r)_{r \geq 1} \in \varprojlim_r M, \text{ such that } x_1 = x,$$

where the projective limit is taken with respect to the multiplication by ℓ maps $M \xrightarrow{\times \ell} M$. Since $x \in M^{\Gamma_n}$, if we set $y_r := x_{r+1}^{\gamma_K^n - 1}$, for all $r \geq 1$, we have $(y_r)_{r \geq 1} \in T_\ell(M)$. By definition, the map ϕ_n sends x to the class of $(y_r)_{r \geq 1}$ in $T_\ell(M)_{\Gamma_n}$. One can easily check that this definition is independent of any choices, ϕ_n is an isomorphism and the diagram in (1) above is indeed commutative. In fact, the maps ϕ_n arise naturally as connecting morphisms in a snake lemma six-term exact sequence.

• **Proving the bijectivity of $\varprojlim_n \pi_n$.** The bijectivity is obviously equivalent to the equality

$$\bigcap_n (1 - \gamma_K^n) T_\ell(M) = 0.$$

Since $M = \bigcup_n M^{\Gamma_n}$ (Γ_K acts continuously on M !) and M has finite co-rank, for all m , there exists an n , such that $M[\ell^m] \subseteq M^{\Gamma_n}$. This fact combined with the isomorphisms ϕ_n implies that

$$\bigcap_n (1 - \gamma_K^n) T_\ell(M) \subseteq \bigcap_m \ell^m T_\ell(M) = 0.$$

□

Corollary 5.7. *For all primes ℓ , we have canonical isomorphisms of $\mathbb{Z}_\ell[[\Gamma_K]]$ -modules*

$$\mathrm{Pic}_T^0(K)^{(\ell)} \xrightarrow{\sim} T_\ell(J_{T_\infty})_{\Gamma_K}, \quad T_\ell(J_{T_\infty}) \xrightarrow{\sim} \mathfrak{X}_{\infty, T}^{(\ell)},$$

where $\mathrm{Pic}_T^0(K)^{(\ell)} := \mathrm{Pic}_T^0(K) \otimes \mathbb{Z}_\ell$ and $\mathfrak{X}_{\infty, T}^{(\ell)} := \mathfrak{X}_{\infty, T} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. (Note that $\mathfrak{X}_{\infty, T}^{(\ell)}$ is the Galois group of the maximal abelian pro- ℓ extension of K_∞ which is unramified away from T_∞ if $\ell \neq p$ and unramified away from T_∞ and at most tamely ramified at T_∞ , if $\ell = p$.)

Proof. Let $M := J_{T_\infty}[\ell^\infty]$. This is a torsion, divisible, \mathbb{Z}_ℓ -module of finite co-rank. (Theorem 3.6 with G trivial, $Z' = X_\infty$, $\mathcal{S}' = \emptyset$ and $\mathcal{T}' = T_\infty$ gives an exact formula for the co-rank.) The continuity of the Γ_K -action on M and the finiteness of M^{Γ_n} are direct consequences of equalities (14) and the finiteness of $\text{Pic}_T^0(K_n)$, for all n . Now, apply part (2) of the preceding Lemma, combined with equalities (14) and isomorphism (15) to conclude the proof of the corollary. \square

Next, we will use Corollary 5.7 to express certain ℓ -adic étale cohomology groups associated to K in terms of the ℓ -adic realizations $T_\ell(J_{T_\infty})$. Let T be as above. Assume that $T \neq \emptyset$. Further, recalling that K is the top field of a Galois extension K/k of Galois group G , let us assume that T is invariant under the G -action on primes in K . In this case, if ℓ is an arbitrary prime, the modules $T_\ell(J_{T_\infty})$, $\mathfrak{X}_{\infty, T}^{(\ell)}$ and $\text{Pic}_T^0(K)^{(\ell)}$ have obvious natural $\mathbb{Z}_\ell[[G_\infty]]$ -module structures and the isomorphisms in Corollary 5.7 preserve those structures.

Now, let ℓ be a prime number with $\ell \neq p$ and let $n \in \mathbb{Z}_{\geq 0}$. As usual, for all $i \in \mathbb{Z}_{\geq 0}$, we denote by $H_{et}^i(O_{K,T}, \mathbb{Z}_\ell(n))$ the i -th étale cohomology group with the coefficients in the ℓ -adic sheaf $\mathbb{Z}_\ell(n)$ for the scheme $\text{Spec}(O_{K,T})$ associated to the subring of T -integers $O_{K,T}$ in K . For the definition and main properties of these ℓ -adic étale cohomology groups, the reader can consult the excellent survey [13]. Functoriality in étale cohomology leads to natural $\mathbb{Z}_\ell[G]$ -module structures on $H_{et}^i(O_{K,T}, \mathbb{Z}_\ell(n))$, for all i and n as above. In what follows, we denote by

$$\kappa_\ell : G_\infty \longrightarrow \text{Aut}(\mu_{\ell^\infty}) \xrightarrow{\sim} \mathbb{Z}_\ell^\times$$

the ℓ -adic cyclotomic character over k , restricted to G_∞ . It is the continuous character which factors through the ℓ -adic character of $\Gamma := \Gamma_k = \text{Gal}(k_\infty/k)$ which sends $\gamma := \gamma_k$ to q . Note that under the canonical identification of G_∞ with a subgroup of $\overline{G} := G \times \Gamma$ made in §4, the character κ_ℓ can be extended to the continuous character $\tilde{\kappa}_\ell$ of \overline{G} which is trivial on G and sends γ to q .

Definition 5.8. Let R be a commutative \mathbb{Z}_ℓ -algebra, M an $R[[G_\infty]]$ -module and $n \in \mathbb{Z}$.

(1) The Tate twist $M(n)$ is the $R[[G_\infty]]$ -module M with the twisted G_∞ -action

$$\sigma * m = \kappa_\ell(\sigma)^n \cdot {}^\sigma m,$$

for all $\sigma \in G_\infty$ and $m \in M$ and the original R -action.

(2) We let $t_n : R[[G_\infty]] \simeq R[[G_\infty]]$ be the R -algebra isomorphism which sends $\sigma \in G_\infty$ to

$$t_n(\sigma) := \kappa_\ell(\sigma)^n \cdot \sigma.$$

Remark 5.9. With notations as above, if M is a finitely generated $R[[G_\infty]]$ -module, then

$$\text{Fit}_{R[[G_\infty]]}(M(n)) = t_{-n}(\text{Fit}_{R[[G_\infty]]}(M)),$$

for all $n \in \mathbb{Z}$. (See [18], Lemma 3.1.)

Definition 5.10. Let M be a $\mathbb{Z}_\ell[[\mathcal{G}]]$ -module, where \mathcal{G} is a profinite, abelian group. We let

$$M^* := \text{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell), \quad M^\vee := \text{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

viewed as $\mathbb{Z}_\ell[[\mathcal{G}]]$ -modules endowed with either the co-variant or the contra-variant \mathcal{G} -actions, depending on the context. The covariant and contra-variant actions are defined by ${}^g f(m) = f(g \cdot m)$ and ${}^g f(m) = f(g^{-1} \cdot m)$, respectively, for all $g \in \mathcal{G}$, $m \in M$ and $f \in M^*$ or $f \in M^\vee$.

Lemma 5.11. With notations as above, we have isomorphisms of $\mathbb{Z}_\ell[G]$ -modules

$$H_{et}^2(O_{K,T}, \mathbb{Z}_\ell(n)) \xrightarrow{\sim} T_\ell(J_{T_\infty})(-n)_{\Gamma_K}^\vee,$$

for all $n \in \mathbb{Z}_{\geq 2}$, where the dual to the right is endowed with the contra-variant G -action.

Proof. The usual Hochschild-Serre spectral sequence argument in Galois cohomology, combined with the well-known finiteness of the groups $H_{et}^i(O_{K,T}, \mathbb{Z}_\ell(n))$, for all $n \in \mathbb{Z}_{\geq 2}$ and all $i \in \mathbb{Z}_{\geq 0}$, leads to isomorphisms of $\mathbb{Z}_\ell[G]$ -modules

$$H_{et}^2(O_{K,T}, \mathbb{Z}_\ell(n)) \xrightarrow{\sim} \mathfrak{X}_{\infty,T}^{(\ell)}(-n)_{\Gamma_K}^\vee,$$

where the dual above is endowed with the contra-variant G -action (see the arguments in [13], pp. 201–203 and pp. 237–238.) The Galois group $\mathfrak{X}_{\infty,T}$ (and implicitly its ℓ -primary component $\mathfrak{X}_{\infty,T}^{(\ell)}$) is endowed with the usual lift-and-conjugation $\text{Gal}(K_\infty/k)$ -action. Now, we combine the isomorphism above with Corollary 5.7 to arrive at the desired result. \square

5.2. The Brumer-Stark and Coates-Sinnott conjectures in characteristic p . With notations as in §4.2, we let S_K and Σ_K denote the sets consisting of all the primes in K sitting above primes in S and Σ , respectively. However, for simplicity, the subscript K will be dropped from the notation whenever convenient (as in $\text{Pic}_\Sigma^0(K)$ or $O_{K,S}$, for example.) Note that, by definition, both S_K and Σ_K are G -invariant.

Conjecture 5.12. (Brumer-Stark) *With notations as above, we have*

$$\Theta_{S,\Sigma}(1) \in \text{Ann}_{\mathbb{Z}[G]}(\text{Pic}_\Sigma^0(K)).$$

Remark 5.13. (a) *For a fixed S and all Σ , so that $(K/k, S, \Sigma)$ satisfy the required properties, the statement above is equivalent to the characteristic p version of the classical Brumer-Stark conjecture for the given S (as stated in [20], Chpt. V). In order to see this, one has to make the following crucial observation. Let us fix S . For every Σ , satisfying the above properties, let*

$$\delta_\Sigma(u) := \prod_{v \in \Sigma} (1 - \sigma_v^{-1} \cdot (qu)^{d_v}).$$

Then the set $\{\delta_\Sigma(1) \mid \Sigma\}$ generates the ideal $\text{Ann}_{\mathbb{Z}[G]}(\mu_K)$, where μ_K denotes the group of roots of unity in K . (This is the exact analogue in characteristic p of Lemma 1.1 in [20], Chpt. V.)

(b) *The conjecture above was proved independently and with different methods by Deligne (see [20], Chpt. V) and Hayes (see [10]). In the next section, we will prove a refined version of this conjecture involving Fitting ideals rather than annihilators of ideal class-groups.*

Conjecture 5.14. (Coates-Sinnott) *With notations as above, we have*

$$\Theta_{S,\Sigma}(q^{n-1}) \in \text{Ann}_{\mathbb{Z}_\ell[G]}(H_{et}^2(O_{K,S}, \mathbb{Z}_\ell(n))),$$

for all primes $\ell \neq p$ and all $n \in \mathbb{Z}_{\geq 2}$.

Remark 5.15. *For a fixed S and all Σ , such that $(K/k, S, \Sigma)$ satisfy the above properties, the statement above is the exact characteristic p analogue of the classical Coates-Sinnott conjecture for that fixed S (as stated in [2]). In order to see this, one needs to make the following observation. If we fix S as above, then, for all $n \in \mathbb{Z}_{\geq 2}$ and all primes $\ell \neq p$, the set $\{\delta_\Sigma(q^{n-1}) \mid \Sigma\}$ generates*

$$\text{Ann}_{\mathbb{Z}_\ell[G]}(H_{et}^1(O_{K,S}, \mathbb{Z}_\ell(n))_{\text{tors}})$$

as a $\mathbb{Z}_\ell[G]$ -ideal, where Σ runs through all finite sets of primes in k , such that S and Σ satisfy the above properties. Since

$$H_{et}^1(O_{K,S}, \mathbb{Z}_\ell(n))_{\text{tors}} = (\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{\Gamma_K}$$

(see [13], pp. 201-203), the statement above is the exact analogue in characteristic p of Lemma 2.3 in [3]. In order to compare the above observation to that in (a) of the last remark, the reader is invited to note that $H_{et}^1(O_{K,S}, \mathbb{Z}_\ell(1))_{\text{tors}} = (\mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{\Gamma_K} = \mu_K^{(\ell)}$, for all $\ell \neq p$, while $\mu_K^{(p)} = 1$.

5.3. Refinements of the Brumer-Stark and Coates-Sinnott conjectures. Next, we prove the promised refinements of the two conjectures stated in the previous section. The notations are the same as above.

Lemma 5.16. *Let M be a $\mathbb{Z}_\ell[[G_\infty]]$ -module, which is free of finite rank as a \mathbb{Z}_ℓ -module and such that M_{Γ_K} is finite. Then, we have an isomorphism of $\mathbb{Z}_\ell[G]$ -modules*

$$(M^*)_{\Gamma_K} \xrightarrow{\sim} (M_{\Gamma_K})^\vee,$$

where both dual modules are viewed with either the co-variant or the contra-variant G_∞ -actions.

Proof. We will prove the statement for the co-variant action and leave the remaining case to the reader. Since M is \mathbb{Z}_ℓ -free of finite rank and M_{Γ_K} is finite, we have $M^{\Gamma_K} = 0$. Consequently, we have an exact sequence of $\mathbb{Z}_\ell[[G_\infty]]$ -modules

$$0 \longrightarrow M \xrightarrow{1-\gamma_K} M \longrightarrow M_{\Gamma_K} \longrightarrow 0.$$

Apply the functor $\text{Hom}_{\mathbb{Z}_\ell}(*, \mathbb{Z}_\ell)$ to the sequence above to obtain an exact sequence

$$0 \longrightarrow M^* \xrightarrow{1-\gamma_K} M^* \longrightarrow \text{Ext}_{\mathbb{Z}_\ell}^1(M_{\Gamma_K}, \mathbb{Z}_\ell) \longrightarrow 0.$$

This shows that we have an isomorphism of $\mathbb{Z}_\ell[G]$ -modules $(M^*)_{\Gamma_K} \simeq \text{Ext}_{\mathbb{Z}_\ell}^1(M_{\Gamma_K}, \mathbb{Z}_\ell)$. Now, one applies the functor $\text{Hom}_{\mathbb{Z}_\ell}(M_{\Gamma_K}, *)$ to the exact sequence

$$0 \longrightarrow \mathbb{Z}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow 0$$

to obtain an isomorphism of $\mathbb{Z}_\ell[G]$ -modules $(M_{\Gamma_K})^\vee \simeq \text{Ext}_{\mathbb{Z}_\ell}^1(M_{\Gamma_K}, \mathbb{Z}_\ell)$. Taking into account the previous isomorphism, this concludes the proof. \square

Corollary 5.17. *The following hold.*

- (1) *For all prime numbers ℓ , we have an isomorphism of $\mathbb{Z}_\ell[G]$ -modules*

$$T_\ell(J_{\Sigma_\infty})^*_{\Gamma_K} \xrightarrow{\sim} \text{Pic}_\Sigma^0(K)^{(\ell)\vee},$$

where the duals are G_∞ -co-variant.

- (2) *For all prime numbers $\ell \neq p$ and all $n \in \mathbb{Z}_{\geq 2}$, we have an isomorphism of $\mathbb{Z}_\ell[G]$ -modules*

$$T_\ell(J_{S_\infty})(-n)^*_{\Gamma_K} \xrightarrow{\sim} H_{et}^2(O_{K,S}, \mathbb{Z}_\ell(n)),$$

where the dual is G_∞ -contra-variant.

Proof. The idea is to apply the Lemma above to $M := T_\ell(J_{\Sigma_\infty})$ and $M := T_\ell(J_{S_\infty})(-n)$. These are free \mathbb{Z}_ℓ -modules of finite rank. We have to show that they also satisfy the finiteness of M_{Γ_K} hypothesis. However, for any M as in the Lemma, M_{Γ_K} is finite if and only if the action of γ_K on the \mathbb{Q}_ℓ -vector space $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} M$ has no fixed points (i.e. it does not have 1 as an eigenvalue.) Since J_{Σ_∞} is an extension of the Jacobian J_{X_∞} by a torus, the Riemann Hypothesis for the smooth, projective curve X_∞ over \mathbb{F} implies that the eigenvalues of the action of γ_K on $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(J_{S_\infty})$ are algebraic integers (independent of ℓ , if $\ell \neq p$) of absolute value r (the torus contribution) or $r^{1/2}$ (the Jacobian contribution.) The same argument applied to $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(J_{S_\infty})(-n)$ leads to

eigenvalues which are algebraic numbers of absolute value either r^{1-n} (torus) or $r^{1/2-n}$ (Jacobian). Since $n \geq 2$, none of these eigenvalues equals 1, so M_{Γ_K} is finite in both cases.

(1) The Lemma above applied to $M := T_\ell(J_{\Sigma_\infty})$, combined with Corollary 5.7 give the following isomorphisms of $\mathbb{Z}_\ell[G]$ -modules.

$$T_\ell(J_{\Sigma_\infty})^*_{\Gamma_K} \simeq T_\ell(J_{\Sigma_\infty})_{\Gamma_K}^\vee \simeq \text{Pic}_\Sigma^0(K)^{(\ell)\vee}.$$

This concludes the proof of (1).

(2) The Lemma above applied to $M := T_\ell(J_{\overline{S}})(-n)$, combined with Lemma 5.11 give the following isomorphisms of $\mathbb{Z}_\ell[G]$ -modules.

$$T_\ell(J_{S_\infty})(-n)^*_{\Gamma_K} \simeq T_\ell(J_{S_\infty})(-n)_{\Gamma_K}^\vee \simeq H_{et}^2(O_{K,S}, \mathbb{Z}_\ell(n)).$$

This concludes the proof of part (2). \square

Theorem 5.18. (“refined Brumer-Stark conjecture”) *Let $\text{Pic}_\Sigma^0(K)^\vee := \text{Hom}_{\mathbb{Z}}(\text{Pic}_\Sigma^0(K), \mathbb{Q}/\mathbb{Z})$, endowed with the co-varient G -action. Then, we have*

$$\Theta_{S,\Sigma}(1) \in \text{Fit}_{\mathbb{Z}[G]}(\text{Pic}_\Sigma^0(K)^\vee).$$

Proof. Since $\Theta_{S,\Sigma}(1) \in \mathbb{Z}[G]$ (see §4.2), it suffices to prove that $\Theta_{S,\Sigma}(1) \in \text{Fit}_{\mathbb{Z}_\ell[G]}(\text{Pic}_\Sigma^0(K)^{(\ell)\vee})$, for all prime numbers ℓ , where $\text{Pic}_\Sigma^0(K)^{(\ell)\vee} := \text{Hom}_{\mathbb{Z}_\ell}(\text{Pic}_\Sigma^0(K)^{(\ell)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ with the co-varient G -action. Now, we bring the 1-motive $M_{S_\infty, \Sigma_\infty}$ into the game (see the paragraph leading to Corollary 4.13.) Let us fix a prime number ℓ . We combine Corollary 4.13 with Corollary 4.2(2) to get

$$(16) \quad \Theta_{S,\Sigma}(\gamma^{-1}) \in \text{Fit}_{\mathbb{Z}_\ell[[G_\infty]]}(T_\ell(M_{S_\infty, \Sigma_\infty})^*),$$

where the \mathbb{Z}_ℓ -dual is endowed with the co-varient G_∞ -action. (Recall that $G_\infty \simeq \Gamma \times H$.) We have the obvious exact sequence of co-varient \mathbb{Z}_ℓ -duals in the category of $\mathbb{Z}_\ell[[G_\infty]]$ -modules

$$0 \longrightarrow (\text{Div}^0(S_\infty) \otimes \mathbb{Z}_\ell)^* \longrightarrow T_\ell(M_{S_\infty, \Sigma_\infty})^* \longrightarrow T_\ell(J_{\Sigma_\infty})^* \longrightarrow 0.$$

The behaviour of Fitting ideals with respect to surjections combined with (16) above leads to

$$\Theta_{S,\Sigma}(\gamma^{-1}) \in \text{Fit}_{\mathbb{Z}_\ell[[G_\infty]]}(T_\ell(J_{\Sigma_\infty})^*).$$

Now, we extend scalars along the surjective ring morphism $\pi : \mathbb{Z}_\ell[[G_\infty]] \twoheadrightarrow \mathbb{Z}_\ell[G]$ and obtain

$$\pi(\Theta_{S,\Sigma}(\gamma^{-1})) \in \text{Fit}_{\mathbb{Z}_\ell[G]}(T_\ell(J_{\Sigma_\infty})^*_{\Gamma_K}).$$

However, the ring morphism π is induced by the composition of group morphisms $G_\infty \subseteq \overline{G} = G \times \Gamma \twoheadrightarrow G$, where $G \twoheadrightarrow G$ is the projection onto the second factor sending γ to 1. Consequently, we have $\pi(\Theta_{S,\Sigma}(\gamma^{-1})) = \Theta_{S,\Sigma}(1)$. Now, we apply Corollary 5.17(1) to conclude the proof of the Theorem. \square

Remark 5.19. *Observe that the theorem above is indeed a refinement of the Brumer-Stark Conjecture. Indeed, since we are dealing with co-varient \mathbb{Z}_ℓ -duals, we have*

$$\text{Fit}_{\mathbb{Z}[G]}(\text{Pic}_\Sigma^0(K)^\vee) \subseteq \text{Ann}_{\mathbb{Z}[G]}(\text{Pic}_\Sigma^0(K)^\vee) = \text{Ann}_{\mathbb{Z}[G]}(\text{Pic}_\Sigma^0(K)),$$

and the inclusion above is strict, in general.

Theorem 5.20. (“refined Coates-Sinnott conjecture”) *For all primes $\ell \neq p$ and all $n \in \mathbb{Z}_{\geq 2}$,*

$$\Theta_{S,\Sigma}(q^{n-1}) \in \text{Fit}_{\mathbb{Z}_\ell[G]}(H_{et}^2(O_{K,S}, \mathbb{Z}_\ell(n))).$$

Proof. Let us fix a prime $\ell \neq p$ and an integer $n \geq 2$. There is a perfect $\mathbb{Z}_\ell[[G_\infty]]$ -linear pairing

$$T_\ell(M_{S_\infty, \Sigma_\infty}) \times T_\ell(M_{\Sigma_\infty, S_\infty}) \longrightarrow \mathbb{Z}_\ell(1),$$

generalizing the Weil pairing on J_{X_∞} (the case $S = \Sigma = \emptyset$) – see §10.2 in [6] for the general theory, but also see our upcoming work [7] for an explicit construction (formula) of such a pairing. The pairing above induces an isomorphism of $\mathbb{Z}_\ell[[G_\infty]]$ -modules

$$T_\ell(M_{S_\infty, \Sigma_\infty}) \xrightarrow{\sim} T_\ell(M_{\Sigma_\infty, S_\infty})^*(1),$$

where the \mathbb{Z}_ℓ -dual is endowed with the contra-variant G_∞ -action. Now, apply Corollary 4.13(2) combined with Remark 5.9 and the above isomorphism to conclude that

$$t_{1-n}(\Theta_{S, \Sigma}(\gamma^{-1})) \in \text{Fit}_{\mathbb{Z}_\ell[[G_\infty]]}(T_\ell(M_{S_\infty, \Sigma_\infty})(n-1)) = \text{Fit}_{\mathbb{Z}_\ell[[G_\infty]]}(T_\ell(M_{\Sigma_\infty, S_\infty})(-n)^*).$$

Above, we used the obvious equality $T_\ell(M_{\Sigma_\infty, S_\infty})(-n)^* = T_\ell(M_{\Sigma_\infty, S_\infty})^*(n)$ of G_∞ -contra-variant \mathbb{Z}_ℓ -duals. Now, we have an obvious exact sequence of $\mathbb{Z}_\ell[[G_\infty]]$ -modules.

$$0 \longrightarrow (\text{Div}^0(\Sigma_\infty) \otimes \mathbb{Z}_\ell)(-n)^* \longrightarrow T_\ell(M_{\Sigma_\infty, S_\infty})(-n)^* \longrightarrow T_\ell(J_{S_\infty})(-n)^* \longrightarrow 0$$

Consequently, as in the proof of the previous theorem, we get the following:

$$t_{1-n}(\Theta_{S, \Sigma}(\gamma^{-1})) \in \text{Fit}_{\mathbb{Z}_\ell[[G_\infty]]}(T_\ell(J_{S_\infty})(-n)^*), \quad \pi(t_{1-n}(\Theta_{S, \Sigma}(\gamma^{-1}))) \in \text{Fit}_{\mathbb{Z}_\ell[G]}(T_\ell(J_{S_\infty})(-n)^*|_{\Gamma_K}).$$

As before, the second relation above is obtained from the first via extension of scalars along the natural surjective ring morphism $\pi : \mathbb{Z}_\ell[[G_\infty]] \rightarrow \mathbb{Z}_\ell[G]$ induced by the composition of group morphisms $G_\infty \subseteq G \times \Gamma \rightarrow G$. It is easy to see that

$$\pi(t_{1-n}(\Theta_{S, \Sigma}(\gamma^{-1}))) = \pi(\Theta_{S, \Sigma}(q^{n-1} \cdot \gamma^{-1})) = \Theta_{S, \Sigma}(q^{n-1})$$

(see comments leading to Definition 5.8.) Now, one combines this equality with the second relation above and the isomorphism of Corollary 5.17(2) to conclude the proof of the Theorem. \square

6. Cartier operators, logarithmic differentials and p -torsion of Picard 1-motives

The main goal of this final section is to show how our Theorem 3.6 implies the main result of Nakajima's paper [16] on Galois p -covers of smooth, connected, projective curves defined over an algebraically closed field of characteristic p . In the process, we place Nakajima's result in the general framework of Picard 1-motives.

6.1. Nakajima's Theorem. In what follows, Z will denote a connected, smooth, projective curve over an algebraically closed field κ of characteristic p . Let $\mathcal{K} := \kappa(Z)$ be the field of κ -rational functions of Z . We denote by Ω_Z the \mathcal{K} -module $\Omega_{\mathcal{K}/\kappa}$ of Kähler differentials (which can also be identified with the generic fiber of the sheaf of differentials on Z .) G will denote a finite group of κ -automorphisms of Z . Obviously, Ω_Z inherits a $\kappa[G]$ -module structure, with the canonical G -action $\sigma(x \cdot dy) = \sigma x \cdot d(\sigma y)$, for all $\sigma \in G$ and all $x, y \in \mathcal{K}$. We let \mathcal{S} denote a G -invariant finite set of closed points in Z and let $\mathcal{S}' := \pi(\mathcal{S})$ (the set of closed points on Z' lying below those in \mathcal{S} .) We take the effective divisor $[\mathcal{S}] = \sum_{P \in \mathcal{S}} P$ on Z and let

$$\Omega_Z(-[\mathcal{S}]) := \{\omega \in \Omega_Z \mid \text{div}(\omega) \geq -[\mathcal{S}]\}$$

be the space of differentials on Z whose divisors are greater than or equal to $-[\mathcal{S}]$.

Definition 6.1. *The Cartier operator $\mathcal{C} : \Omega_Z \rightarrow \Omega_Z$ is the unique \mathbb{F}_p -linear map, satisfying the following properties.*

- (1) $\mathcal{C}(x^p \omega) = x \mathcal{C}(\omega)$, $\forall x \in \mathcal{K}$ and $\forall \omega \in \Omega_Z$, i.e. \mathcal{C} is p^{-1} – linear.

- (2) $\mathcal{C}(df) = 0, \forall f \in \mathcal{K}.$
- (3) $\mathcal{C}(x^{p-1}dx) = dx, \forall x \in \mathcal{K}.$

The reader may consult [16] and [19] for the existence and uniqueness of \mathcal{C} . The uniqueness of \mathcal{C} implies right away that \mathcal{C} is an $\mathbb{F}_p[G]$ -linear map. Also, since \mathcal{S} is G -invariant, Lemma 2.1 in [19] implies that \mathcal{C} induces an $\mathbb{F}_p[G]$ -linear map

$$\mathcal{C} : \Omega_Z(-[\mathcal{S}]) \longrightarrow \Omega_Z(-[\mathcal{S}]).$$

Definition 6.2. Let $\Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1}$ be the \mathbb{F}_p -vector subspace of $\Omega_Z(-[\mathcal{S}])$ consisting of all the differentials ω which are fixed by the Cartier operator \mathcal{C} . Let $\Omega_Z(-[\mathcal{S}])^s$ be the κ -vector subspace of $\Omega_Z(-[\mathcal{S}])$ spanned by $\Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1}$. $\Omega_Z(-[\mathcal{S}])^s$ is called the semi-simple part of $\Omega_Z(-[\mathcal{S}])$ with respect to the p^{-1} -linear operator \mathcal{C} .

Remark 6.3. Obviously, if G is a group of automorphisms of Z fixing \mathcal{S} , then $\Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1}$ is an $\mathbb{F}_p[G]$ -module and $\Omega_Z(-[\mathcal{S}])^s$ is a $\kappa[G]$ -module. Also, it turns out that one has a direct sum decomposition in the category of $\kappa[G]$ -modules

$$\Omega_Z(-[\mathcal{S}]) = \Omega_Z(-[\mathcal{S}])^s \oplus \Omega_Z(-[\mathcal{S}])^n,$$

where $\Omega_Z(-[\mathcal{S}])^n$ is the nilpotent space associated to \mathcal{C} , consisting of all differential forms $\omega \in \Omega_Z(-[\mathcal{S}])$ which are killed by a power of \mathcal{C} (see Theorem 2.2 in [19].)

Theorem 6.4. (Nakajima) Assume that $\pi : Z \mapsto Z'$ is a G -Galois cover of connected, smooth, projective curves over κ , where G is a finite p -group. Assume that \mathcal{S} is a finite, non-empty, G -equivariant set of closed points on Z , containing the ramification locus for π . Then $\Omega_Z(-[\mathcal{S}])^s$ is a free $\kappa[G]$ -module whose rank is given by

$$\text{rank}_{\kappa[G]} \Omega_Z(-[\mathcal{S}])^s = (\gamma_{Z'} - 1 + |\mathcal{S}'|).$$

Proof. See [16], Theorem 1, p. 561. □

6.2. Nakajima's theorem in the language of p -torsion of Picard 1-motives. For the moment, let us assume that $Z \rightarrow Z'$ is a G -Galois cover of smooth, connected, projective curves defined over κ and that \mathcal{S} is a G -invariant, finite, non-empty set of closed points on Z , containing the ramification locus of the cover. Next, we make a connection between the $\kappa[G]$ -module $\Omega_Z(-[\mathcal{S}])^s$ and the $\mathbb{F}_p[G]$ -module $\mathcal{M}_{\mathcal{S}, \emptyset}[p]$ of p -torsion points of the Picard 1-motive $\mathcal{M}_{\mathcal{S}, \emptyset}$ associated to the data $(Z, \kappa, \mathcal{S}, \emptyset)$. As a consequence, we show how Nakajima's Theorem 6.4 is in fact equivalent to a particular case of our Theorem 3.6.

Lemma 6.5. The following hold true.

- (1) Let $\omega \in \Omega_Z(-[\mathcal{S}])$. Then $\omega \in \Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1}$ if and only if there exists $f \in \mathcal{K}^\times$, such that $\omega = \frac{df}{f}$.
- (2) Let $f \in \mathcal{K}^\times$. Then $\frac{df}{f} \in \Omega_Z(-[\mathcal{S}])$ if and only if $f \in \mathcal{K}_{\mathcal{S}, \emptyset}^{(p)}$.
- (3) There exists an isomorphism of $\mathbb{F}_p[G]$ -modules

$$\mathcal{K}_{\mathcal{S}, \emptyset}^{(p)} / \mathcal{K}_{\emptyset}^{\times p} \simeq \Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1}$$

given by $\widehat{f} \mapsto \frac{df}{f}$, for all $f \in \mathcal{K}_{\mathcal{S}, \emptyset}^{(p)}$.

- (4) There exists an isomorphism of $\mathbb{F}_p[G]$ -modules

$$\mathcal{M}_{\mathcal{S}, \emptyset}[p] \simeq \Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1}$$

for all G -invariant, finite (possibly empty) sets \mathcal{T} of closed points on Z , disjoint from \mathcal{S} .

Proof. For (1) see ‘‘FACT’’ in [19], p. 4, for example. The proof of (2) can also be found in [19], but since it is straightforward, for the convenience of the reader we sketch it next. Let $f \in \mathcal{K}^\times$ and let P be a closed point on Z . Let $t \in \mathcal{K}$ be a uniformizer at P and let $\mathcal{K}_P \simeq \kappa((t))$ be the completion of \mathcal{K} at P . Assume that $\text{ord}_P(f) = n$ and $f = a_n t^n + a_{n+1} t^{n+1} + \dots$ in $\kappa((t))$, with $a_n, a_{n+1}, \dots \in \kappa$ and $a_n \neq 0$. Then the image of $\omega := \frac{df}{f}$ via the natural embedding $\Omega_Z \hookrightarrow \Omega_{\kappa((t))/\kappa} \simeq \kappa((t))dt$ is given by

$$\frac{df}{f} = \left(\frac{n}{t} + b_0 + b_1 t + \dots \right) dt,$$

with b_0, b_1, \dots in κ . This shows that $\frac{df}{f} \in \Omega_Z(-[\mathcal{S}])$ if and only if $p \mid \text{ord}_P(f)$, for all $P \notin \mathcal{S}$, i.e. if and only if $f \in \mathcal{K}_{\mathcal{S}, \emptyset}^{(p)}$.

Now, by first noting that $\frac{d(fg)}{fg} = \frac{df}{f} + \frac{dg}{g}$, for all $f, g \in \mathcal{K}^\times$, one easily sees that (1) and (2) imply that we have a surjective $\mathbb{Z}[G]$ -module morphism $\mathcal{K}_{\mathcal{S}, \emptyset}^{(p)} \twoheadrightarrow \Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1}$, given by $f \mapsto \frac{df}{f}$. The kernel of this morphism consists of all $f \in \mathcal{K}_{\mathcal{S}, \emptyset}^{(p)}$, such that $df = 0$. This kernel is easily seen to equal $\mathcal{K}_\emptyset^{\times p}$, by picking a uniformizer t at an arbitrary closed point P on Z and (canonically) embedding $\Omega_Z \simeq \mathcal{K}dt$ into $\Omega_{\kappa((t))/\kappa} \simeq \kappa((t))dt$, as before. This concludes the proof of (3).

In order to prove part (4), fix \mathcal{T} as above. If one combines Remark 2.7 and Proposition 2.9 with part (3) of our Lemma, one obtains the desired isomorphism. \square

Lemma 6.6. *The canonical κ -linear map*

$$\Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1} \otimes_{\mathbb{F}_p} \kappa \longrightarrow \Omega_Z(-[\mathcal{S}])^s, \quad \omega \otimes x \mapsto x\omega$$

induces an isomorphism of $\kappa[G]$ -modules.

Proof. It suffices to show that if $\{\omega_1, \dots, \omega_n\}$ are \mathbb{F}_p -linearly independent elements of $\Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1}$, then they remain κ -linearly independent in $\Omega_Z(-[\mathcal{S}])$. Assume that this is not true. Without loss of generality, one may assume that $\{\omega_1, \dots, \omega_{n-1}\}$ are κ -linearly independent, but

$$\omega_n = x_1 \cdot \omega_1 + \dots + x_{n-1} \cdot \omega_{n-1},$$

for some $x_1, \dots, x_{n-1} \in \kappa$. Now, apply \mathcal{C} to the equality above. We obtain

$$\omega_n = x_1^{1/p} \cdot \omega_1 + \dots + x_{n-1}^{1/p} \cdot \omega_{n-1}.$$

This implies (via the assumed κ -linear independence of $\omega_1, \dots, \omega_{n-1}$) that

$$x_1 = x_1^{1/p}, \dots, x_{n-1} = x_{n-1}^{1/p}.$$

Consequently, $x_1, \dots, x_{n-1} \in \mathbb{F}_p$, which contradicts the \mathbb{F}_p -linear independence of $\{\omega_1, \dots, \omega_n\}$. \square

Next, we state and prove the main result of this section.

Proposition 6.7. *Let κ be an algebraically closed field of characteristic p . Let $Z \rightarrow Z'$ be a G -Galois cover of connected, smooth, projective curves defined over κ . Let \mathcal{S} and \mathcal{T} be G -equivariant, disjoint, finite sets of closed points on Z , such that \mathcal{S} is non-empty and contains the ramified locus of the cover. Assume that G is a p -group. Then, the following are equivalent.*

(1) **(Nakajima’s Theorem)** $\Omega_Z(-[\mathcal{S}])^s$ is a free $\kappa[G]$ -module whose rank is given by

$$\text{rank}_{\kappa[G]} \Omega_Z(-[\mathcal{S}])^s = (\gamma_{Z'} - 1 + |\mathcal{S}'|).$$

(2) **(Theorem 3.6 for $\ell = p$)** $\mathcal{M}_{\mathcal{S}, \mathcal{T}}[p]$ is a free $\mathbb{F}_p[G]$ -module whose rank is given by

$$\text{rank}_{\mathbb{F}_p[G]} \mathcal{M}_{\mathcal{S}, \mathcal{T}}[p] = (\gamma_{Z'} - 1 + |\mathcal{S}'|).$$

Proof. Lemma 6.6 gives an isomorphism of $\kappa[G]$ -modules

$$\Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1} \otimes_{\mathbb{F}_p[G]} \kappa[G] \xrightarrow{\sim} \Omega_Z(-[\mathcal{S}])^s, \quad \omega \otimes x \mapsto x\omega.$$

Combine this isomorphism with the fact that $\kappa[G]$ is a faithfully flat $\mathbb{F}_p[G]$ -algebra to conclude that (1) above is equivalent to the statement

$$(1') \quad \Omega_Z(-[\mathcal{S}])^{\mathcal{C}=1} \text{ is a free } \mathbb{F}_p[G]\text{-module of rank } (\gamma_{Z'} - 1 + |\mathcal{S}'|).$$

Now, use the isomorphism in Lemma 6.5 (4) to conclude that statement (1') above is equivalent to statement (2) in our Proposition. \square

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